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Dehn twists and Lagrangian spherical manifolds

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Abstract

We study Dehn twists along Lagrangian submanifolds that are finite free quotients of spheres. We describe the induced auto-equivalences to the derived Fukaya category and explain their relations to mirror symmetry.

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Contents

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1 Introduction

In his early groundbreaking papers [1,2], Seidel studied the Dehn twist along a Lagrangian sphere and its induced auto-equivalence on the derived Fukaya category. There are often no automorphism of the mirror which induces such an auto-equivalence [3]. It turns out that this auto-equivalence of the mirror, called a spherical twist, can be described purely categorically and there are a lot of generalizations of spherical twists and spherical objects, including $\mathbb{P}$-twist, family twist [4], etc.

Many of these generalizations are also motivated by the corresponding symplectomorphisms associated to Lagrangian objects. For example, Lagrangian Dehn twists along spheres can be easily generalized to submanifolds whose geodesics are all closed with the same period. When the Lagrangian submanifold is a complex projective space, Huybrechts and Thomas conjectured that the resulting symplectomorphism induces a $\mathbb{P}$-twist in the Fukaya category [5]. However, in most cases, this is still conjectural. Recently, the authors made progress on Huybrechtz–Thomas conjecture by showing that Dehn twists along Lagrangian projective spaces yields a mapping cone operation predicted in the form of $\mathbb{P}$-twists on the Fukaya category. In general, it is still very difficult to compute the auto-equivalence of a given symplectomorphism.

In this paper, we investigate a new type of Dehn twist and its associated auto-equivalences.

Question 1.1 *On a Fukaya category, what is the induced auto-equivalence of the Dehn twist along a* ***spherical Lagrangian****, i.e. a Lagrangian submanifold P whose universal cover is S^n?*

A particularly interesting feature of these twist auto-equivalences, which distinguishes this question from all previous twist auto-equivalences, is its sensitivity to the characteristic of the ground field.

Consider the basic example of $P = \mathbb{RP}^n$. In characteristic zero, P is a spherical object in the Fukaya category. In Corollary 1.3 we show that the induced auto-equivalence is a composition of two spherical twists. However, when $char = 2$, P becomes a $\mathbb{P}^n$-object and the auto-equivalence is a $\mathbb{P}$-twist as defined in [5]. Indeed, given a spherical Lagrangian that is a more complicated quotient of a sphere, its twist auto-equivalence decomposes into a composition of spherical twists in characteristic zero, but when one considers ground field of non-zero characteristics, such twists yield an entire family of previously unknown auto-equivalences. We hope this result contributes to the increasing interests in studying derived categories and Fukaya categories of finite characteristics.

To explain our result, let $\mathbb{K}$ be a field of any characteristic and $\Gamma \subset SO(n+1)$ be a finite subgroup for which there exists $\widetilde{\Gamma} \subset Spin(n+1)$ such that the covering homomorphism $Spin(n+1) \to SO(n+1)$ restricts to an isomorphism $\widetilde{\Gamma} \simeq \Gamma$. Let P be a Lagrangian submanifold that is diffeomorphic to S^n/Γ in a Liouville manifold (M, ω) with $2c_1(M, \omega) = 0$. Pick a Weinstein neighborhood U of P and take the universal cover $\mathbf{U}$ of U. The preimage of P is a Lagrangian sphere $\mathbf{P}$ in $\mathbf{U}$. We can pick a parametrization to identify $\mathbf{P}$ with the unit sphere in $\mathbb{R}^{n+1}$, and the deck transformation with $\Gamma \subset SO(n+1)$. Then we can define the Dehn twist $\tau_{\mathbf{P}}$ along $\mathbf{P}$ in $\mathbf{U}$. Since $\tau_{\mathbf{P}}$ is defined by geodesic flow with respect to the round metric on $\mathbf{P}$ and the antipodal map lies in the center of $SO(n+1)$, $\tau_{\mathbf{P}}$ is Γ-equivariant and descends to a symplectomorphism τ_P in U. We call τ_P the *Dehn twist along* P.

We equip P with the induced spin structure from S^n and with the universal local system E corresponding to the canonical representation of $\Gamma := \pi_1(P)$ to $\mathbb{K}[\Gamma]$. The pair (P, E) defines an object $\mathcal{P}$ in the compact Fukaya category $\mathcal{F}$. For any Lagrangian brane (i.e. an exact Lagrangian submanifold with a choice of grading, spin structure and local system) $\mathcal{E}$ in (M, ω), we have a left Γ-module structure on $hom_{\mathcal{F}}(\mathcal{E}, \mathcal{P})$ and a right Γ-module structure on $hom_{\mathcal{F}}(\mathcal{P}, \mathcal{E})$. Our main result is

Theorem 1.2 *Let (M^{2n}, ω) be a Liouville manifold with $2c_1(M) = 0$ and $n \geqslant 3$. For any exact Lagrangian brane $\mathcal{E} \in \mathcal{F}$, there is a quasi-isomorphism of the objects*

$$\tau_P(\mathcal{E}) \simeq Cone(hom_{\mathcal{F}}(\mathcal{P}, \mathcal{E}) \otimes_\Gamma \mathcal{P} \xrightarrow{ev_\Gamma} \mathcal{E}) \tag{1.1}$$

in $\mathcal{F}^{\mathrm{perf}}$, where ev_Γ is the equivariant evaluation map, $Cone$ is the A_∞ mapping cone and $\mathcal{F}^{\mathrm{perf}}$ is the category of perfect A_∞ right $\mathcal{F}$ modules.

On cohomological level, Theorem 1.2 implies that for any Lagrangian branes $\mathcal{E}_0, \mathcal{E}_1 \in \mathcal{F}$, there is a long exact sequence between the Floer cohomology groups

$$\begin{aligned} &HF^{*-1}(\mathcal{E}_0, \tau_P(\mathcal{E}_1)) \to H^*(CF(\mathcal{P}, \mathcal{E}_1) \otimes_\Gamma CF(\mathcal{E}_0, \mathcal{P})) \\ &\quad \to HF^*(\mathcal{E}_0, \mathcal{E}_1) \to HF^*(\mathcal{E}_0, \tau_P(\mathcal{E}_1)) \end{aligned}$$

It is natural to speculate that (1.1) holds on the functor level, i.e. $\tau_P \cong Cone(\mathcal{P} \otimes_\Gamma \mathcal{P} \xrightarrow{ev_\Gamma} Id)$. Theorem 1.2 only shows this is true on the object level but doesn't contain information on the morphisms or their compositions.

For the precise definition of $\mathcal{P}$ and the equivariant evaluation map ev_Γ, readers are referred to Sect. 2.5. Roughly, $\mathcal{P}$ should be thought of as a homological-algebraic incarnation of the immersed Lagrangian represented by the universal cover $S^n \to P$. The equivariant evaluation is an adaption of the usual evaluation in this context. Our main theorem has the following consequence when $P = \mathbb{RP}^n$.

Corollary 1.3 *If P is diffeomorphic to $\mathbb{RP}^n$ for $n = 4k - 1$ and* $\mathrm{char}(\mathbb{K}) \neq 2$*, then there are two orthogonal spherical objects $P_1, P_2 \in \mathcal{F}$ coming from equipping P with different rank one local systems, and $\tau_P(\mathcal{E}) \cong \tau_{P_1}\tau_{P_2}(\mathcal{E})$.*

If $P = \mathbb{RP}^n$ for n odd and $\mathrm{char}(\mathbb{K}) = 2$*, then P is a $\mathbb{P}$-object and $\tau_P(\mathcal{E})$ is quasi-isomorphic to applying $\mathbb{P}$-twist to $\mathcal{E}$ along P.*

Remark 1.4 We would like to remark that Theorem 1.2 has the following reformulation. Under the assumption of Theorem 1.2, there is a spherical functor (see [6] for more about spherical functors)

$$\mathcal{S} : \mathbb{K}[\Gamma]^{\mathrm{perf}} \to \mathcal{F}^{\mathrm{perf}}$$

given by $V \mapsto V \otimes_\Gamma \mathcal{P}$ (see Sect. 2.5). Moreover, for any $\mathcal{E} \in \mathcal{F}$, we have

$$\tau_P(\mathcal{E}) \simeq \mathcal{T}_{\mathcal{S}}(\mathcal{E})$$

in $\mathcal{F}^{\mathrm{perf}}$, where $\mathcal{T}_{\mathcal{S}}$ is the twist auto-equivalence of $\mathcal{F}^{\mathrm{perf}}$ associated to $\mathcal{S}$.

Examples and outlooks

The current paper is focused on the foundations of the theory of twist auto-equivalences associated to τ_P and is the starting point of a series of works investigating examples involving Lagrangian spherical space forms. Although we will not discuss these examples in depth, we give an overview of several forthcoming projects to give the readers an idea on the potential applications of the twist formula and its relations to existing works.

- In an upcoming paper [7], the first author and Ruddat construct Lagrangian embeddings of graph manifolds (e.g. spherical space forms) systematically in some Calabi–Yau 3-folds using toric degenerations and tropical curves. Previous constructions in smooth toric varieties and open Calabi–Yau manifolds using tropical curves can be found in [8] and [9], respectively.
  Lagrangian spherical space forms have been studied in some physics literature (see e.g. [10]) and Dehn twists along them can be realized as the monodromy around a special point in the complex moduli. Our study in this paper can be viewed as the mirror-dual of the intensive study of monodromy actions on the derived category of coherent sheaves in the stringy Kähler moduli space ([4,11–14], etc).
- Hong, Lau and the first author study the local mirror symmetry in all characteristics in a subsequent paper [15] when two lens spaces P, P' are plumbed together. In this case, the lens spaces can be identified with fat spherical objects in the sense of Toda [16] in certain characteristics. This shows that Dehn twists along lens spaces are mirror to fat spherical twists in this case.
  Independently, in the upcoming work [17], Evans, Smith and Wemyss relate Fukaya categories of plumbings of 3-spheres along a circle with derived categories of sheaves on local Calabi–Yau 3-folds containing two floppable curves. Both Lens space twists and fat spherical twists naturally arise in specific characteristics in that setting.
- In principle, Theorem 1.2 can be deduced from the Lagrangian cobordism formalism [18–20]. There are several additional ingredients that need to be taken

into account, though. In the most naive attempt, similar to [21], one needs to use an immersed Lagrangian cobordism that does not have clean self-intersections, which would not even have Gromov compactness on holomorphic disks. A fix could be to generalize the *bottleneck immersed cobordism* [21] to the categorical level, which should yield the desired mapping cone relation.
Note that this bottleneck immersed formalism is different from the ongoing work of Biran and Cornea on the immersed Lagrangian cobordism, but their framework should also enter the picture. We have not adopted this approach since the relevant tools are still under construction, but such an alternative approach should be of independent interest and yields a functor level statement mentioned below Theorem 1.2.

- Another possible approach to Theorem 1.2, explained to us by Ivan Smith, is to realize the Dehn twists as the monodromy in certain symplectic fibrations and apply the Ma'u–Wehrheim–Woodward quilt formalism [22]. This point of view is particularly well-adapted to the case of $P = \mathbb{RP}^n$. In this case, τ_P can be realized as the monodromy of a Morse–Bott Lefschetz fibration, and one could try using the techniques developed by Wehrheim and Woodward in [23]. When P is a general spherical space form, the symplectic fibration is no longer Morse–Bott and more technicalities will be involved. Carrying out this approach would be of independent interest, and it provides another possible approach to the functor version of Theorem 1.2.

The examples mentioned above mostly involve lens spaces where the group Γ is a cyclic group. The algebro geometric counterparts of Dehn twists along more general spherical space forms such as Chiang Lagrangians will be investigated in future works.

Sketch of the proof

The proof of Theorem 1.2 occupies the rest of this paper. Here we give a roadmap of the proof, along with a summary of each section in the paper.

In Sect. 2, we review Lagrangian objects with local systems in the Fukaya categories. When the underlying Lagrangian has finite fundamental group, we introduced its universal local system and regard it as the immersed object coming from the universal cover the the Lagrangian. This gives the object $\mathcal{P}$ in Theorem 1.2 when the underlying Lagrangian is a finite quotient of S^n. We also define the equivariant evaluation map in (1.1).

Section 3 contains most technical tools we will need from symplectic field theory and gradings, where the main new ingredient is an adaption of [24–27], which shows the regularity of various holomorphic curves that we will encounter later.

In Sect. 4, we apply symplectic field theory to understand the holomorphic curves contributing to the Floer differentials, and prove a cohomological version of Theorem 1.2, that is, Proposition 4.1. To achieve this, we first give an identification of generators on both sides by geometrically identifying the intersections, then apply neck-stretching around $\mathcal{P}$ to holomorphic curves (triangles and strips) involved in both sides of (5.2). We prove, by studying the resulting configuration, that the limiting curves in the complement of U are identical for the corresponding differentials under our earlier identification of the generators. In other words, we show that the two cochain complexes are indeed *isomorphic* when the neck is stretched long enough.

In Sect. 5, we prove the categorical version by constructing an appropriate degree zero cocycle between the objects on the two sides of (1.1), which induces the quasi-isomorphism in (1.1) (and hence finish the proof of Theorem 1.2). This cocycle $c_{\mathcal{D}}$ lives in $\mathcal{D}$, which is defined in (5.3). Geometrically, we perturb the object L_1 to a nearby copy L_1' and consider its intersection with the union of L_1 and P, which consist the generators of $\mathcal{D}$. There is an intersection between L_1 and L_1' that represents that fundamental cycle e_L, which is intact after the Dehn twist because it is away from the support. We pursue the naive idea that, this intersection (denoted as $t_{\mathcal{D}}$ when considered as a cochain in $\mathcal{D}$) should be the cocycle we are looking for in $\mathcal{D}$. Unfortunately, $t_{\mathcal{D}}$ is not closed. However, we show that its differential has the form of an upper triangular matrix in Proposition 5.8. To supplement this fact, we computed the differentials from degree zero cochains that that supported at intersections between $L_1' \cap P$. We then correct $t_{\mathcal{D}}$ by considering the multiplications of terms from the term $CF(\mathcal{P}, \mathcal{E}^1) \otimes_\Gamma CF(\tau_P((\mathcal{E}^1)'), \mathcal{P})$ and prove that one can find a cocycle $c_{\mathcal{D}}$ in the form of Proposition 5.16. A further study in the multiplications involving $c_{\mathcal{D}}$ shows it indeed induces a quasi-isomorphism (1.1), hence proving Theorem 1.2. Again, the study of relevant μ^k-multiplications are based on SFT and neck-stretching. The orientation is discussed in the "Appendix".

Some notations.

- Γ is a finite group.
- P is a Lagrangian submanifold diffeomorphic to S^n/Γ for some $\Gamma \subset SO(n+1)$ and P is spin (see Remark 2.9).
- $\mathbf{L}$ is the universal cover of L and $\pi : \mathbf{L} \to L$ (or $\pi : T^*\mathbf{L} \to T^*L$) is the covering map. In particular, $\mathbf{P}$ is the universal cover of P.
- $\mathbf{p} \in \mathbf{L}$ is a lift of $p \in L$.
- $c_{\mathbf{p},\mathbf{q}}$ is the geometric intersection $\pi(T^*_{\mathbf{p}}\mathbf{P} \cap \tau_{\mathbf{P}}(T^*_{\mathbf{q}}\mathbf{P})) \in T^*P$ [see (4.5)].
- $\mathcal{P}$ denotes P equipped with the universal local system, and $\mathcal{E}$ is a Lagrangian equipped with some local system.

Standing assumption (M, ω) is a Liouville manifold with $2c_1(M, \omega) = 0$, and a fixed choice of a trivialization of $(\Lambda^{top}_{\mathbb{C}} T^*M)^{\otimes 2}$ is chosen. All Lagrangians are equipped with a $\mathbb{Z}$-grading and a spin structure.

2 Floer theory with local systems

In this section, we discuss the Floer theory for Lagrangians with local systems in the spirit of [28]. In Sect. 2.1, we review the definition of the Fukaya category. Universal local systems are introduced in Sects. 2.2 and 2.3, accompanied with some algebraic results surrounding this notion. These results might be known to some very experts but were not found in the literature to the best of the authors' knowledge. We have intentionally spelled them out in the most explicit way in our capability, with in mind its comparison with immersed Floer theory, from which some readers could find independent interest. These preliminary results enable us to explain the object $\mathcal{P}$ in

Sect. 2.4 and the evaluation map in Sect. 2.5. Discussions about gradings can be found in [29], [2, Section 11,12].

2.1 Fukaya categories with local systems

Let L be a closed exact Lagrangian submanifold in (M, ω) with a base point $o_L \in L$. Let E be a finite rank local system on L with a flat connection ∇. For a path $c : [0, 1] \to L$, we denote the parallel transport from $E_{c(0)}$ to $E_{c(1)}$ along c with respect to the connection ∇.

$$I_c : E_{c(0)} \to E_{c(1)}.$$

We use the monodromy action from $\Gamma := \pi_1(L)$ to E_{o_L} to endow (E, ∇) a *right* Γ-module structure. More explicitly, for $y \in E_{o_L}$ and $g \in \Gamma$, the right action is given by

$$\rho : \Gamma \to End(E_{o_L}) \tag{2.1}$$

$$g \mapsto (y \mapsto I_g y). \tag{2.2}$$

In particular, $(yg)h = I_h(I_g y) = I_{g*h} y = y(g * h)$, where $*$ stands for concatenation of paths (i.e. g goes first). We use $\mathcal{E}$ to denote the triple (L, E, ∇). For a Hamiltonian diffeomorphism $\phi \in Ham(M, \omega)$, we define $\phi(\mathcal{E}) := (\phi(L), \phi_* E, \phi_* \nabla)$.

Let $\mathcal{E}^i := (L_i, E^i, \nabla^i)$ for $i = 0, 1$. A family of *compactly supported* Hamiltonian functions $H = (H_t)_{t\in[0,1]}$ is called (L_0, L_1)-**admissible** if

$$\phi^H(L_0) \pitchfork L_1 \tag{2.3}$$

where ϕ^H is the time one flow of the Hamiltonian vector field $X_H = (X_{H_t})_{t\in[0,1]}$. Let $\mathcal{X}(L_0, L_1)$ be the set of H-Hamiltonian chord from L_0 to L_1 (i.e. $x : [0, 1] \to M$ such that $\dot{x}(t) = X_H(x(t))$, $x(0) \in L_0$ and $x(1) \in L_1$). The Floer cochain complex between $\mathcal{E}^0$ and $\mathcal{E}^1$ is defined by

$$CF(\mathcal{E}^0, \mathcal{E}^1) := \bigoplus_{x\in\mathcal{X}(L_0,L_1)} Hom_{\mathbb{K}}(E^0_{x(0)}, E^1_{x(1)}) \tag{2.4}$$

Now, we want to introduce some notations to define the differential for $CF(\mathcal{E}^0, \mathcal{E}^1)$ as well as the A_∞-structure for a collection of Lagrangians with local systems.

Let $\mathcal{R}^{d+1}$ be the space of holomorphic disks with $d + 1$ boundary punctures. For each $S \in \mathcal{R}^{d+1}$, one of the boundary punctures is distinguished and it is denoted by ξ_0. The other boundary punctures are ordered counterclockwisely along the boundary and are denoted by $\xi_1, \ldots, \xi_d$, respectively. We denote the boundary component of S from ξ_j to ξ_{j+1} by $\partial_j S$ for $j = 0, \ldots, d - 1$. The boundary component from ξ_d to ξ_0 is denoted by $\partial_d S$. For $j = 1, \ldots, d$, we pick an outgoing/positive strip-like end for ξ_j, which is a holomorphic embedding $\epsilon_j : \mathbb{R}_{\geqslant 0} \times [0, 1] \to S$ such that

$$\begin{cases} \epsilon_j(s, 0) \in \partial_{j-1} S \\ \epsilon_j(s, 1) \in \partial_j S \\ \lim_{s\to\infty} \epsilon_j(s, t) = \xi_j \end{cases} \tag{2.5}$$

We also pick an incoming/negative strip-like end for ξ_0, which is a holomorphic embedding $\epsilon_0 : \mathbb{R}_{\leqslant 0} \times [0,1] \to S$ such that

$$\begin{cases} \epsilon_0(s,0) \in \partial_0 S \\ \epsilon_0(s,1) \in \partial_d S \\ \lim\limits_{s \to -\infty} \epsilon_0(s,t) = \xi_0 \end{cases} \tag{2.6}$$

The strip-like ends are assumed to have pairwise disjoint image and they vary smoothly with respect to S in $\mathcal{R}^{d+1}$.

Let $\{\mathcal{E}^j\}_{j=0}^d$ be a finite collection of Lagrangians with local systems. For $j = 1, \ldots, d$, let H_j be a (L_{j-1}, L_j)-admissible Hamiltonian [see (2.3)]. We also pick a (L_0, L_d)-admissible Hamiltonian H_0. For each $S \in \mathcal{R}^{d+1}$ and each collection $\{H_j\}_{j=0}^d$, we pick a $C^\infty_{cpt}(M)$-valued one-form $K \in \Omega^1(S, C^\infty_{cpt}(M))$. Let $X_K \in \Omega^1(S, C^\infty(M, TM))$ be the corresponding Hamiltonian-vector-field-valued one-form. We require that

$$\begin{cases} \epsilon_j^* X_K = X_{H_j} dt \\ X_K|_{\partial_j S} = 0 \end{cases}. \tag{2.7}$$

When $d = 1$, we assume that $K(s,t) = H_{0,t} = H_{1,t}$ for all $(s,t) \in \mathbb{R} \times [0,1]$. We also assume that K varies smoothly with respect to S and is consistent with respect to gluing near boundary strata of the Deligne–Mumford–Stasheff compactification of $\mathcal{R}^{d+1}$.

Let J^M be an ω-compatible almost complex structure that is cylindrical over the infinite end of M (see Definition 3.1). Let $\mathcal{J}(M,\omega)$ be the space of ω-compatible almost complex structures J such that $J = J^M$ outside a compact set. For $j = 0, \ldots, d$, let $J_j = (J_{j,t})_{t \in [0,1]}$ be a family such that $J_{j,t} \in \mathcal{J}(M,\omega)$ for all t. For each $S \in \mathcal{R}^{d+1}$ and each collection $\{J_j\}_{j=0}^d$, we pick a domain-dependent ω-compatible almost complex structure $J = (J_z)_{z \in S}$ such that

$$\begin{cases} J_z \in \mathcal{J}(M,\omega) & \text{for all } z \\ J \circ \epsilon_j(s,t) = J_{j,t} & \text{for all } j, s, t \end{cases} \tag{2.8}$$

When $d = 1$, we require that $J = (J_{s,t})_{(s,t) \in \mathbb{R} \times [0,1]} = (J_t)_{t \in [0,1]}$ is independent of the s-direction. We assume that J varies smoothly with respect to S in $\mathcal{R}^{d+1}$ and is consistent with respect to gluing near boundary strata of the Deligne–Mumford–Stasheff compactification of $\mathcal{R}^{d+1}$.

Let $x_j \in \mathcal{X}(L_{j-1}, L_j)$ for $j = 1, \ldots, d$ and $x_0 \in \mathcal{X}(L_0, L_d)$. For $d > 1$, we define $\mathcal{M}^{K,J}(x_0; x_d, \ldots, x_1)$ to be the space of smooth maps $u : S \to M$ such that

$$\begin{cases} S \in \mathcal{R}^{d+1} & \\ (du - X_K)^{0,1} = 0 & \text{with respect to } (J_z)_{u(z)} \\ u(\partial_j S) \subset L_j & \text{for all } j \\ \lim\limits_{s \to \pm\infty} u(\epsilon_j(s,t)) = x_j(t) & \text{for all } j \end{cases} \tag{2.9}$$

When $d = 1$, we define $\mathcal{M}^{K,J}(x_0; x_1)$ to be the corresponding space of maps after modulo the $\mathbb{R}$ action by translation in the s-coordinate. For simplicity, we may use $\mathcal{M}(x_0; x_d, \ldots, x_1)$ to denote $\mathcal{M}^{K,J}(x_0; x_d, \ldots, x_1)$ for an appropriate choice of (K, J).

Remark 2.1 In Sect. 3, we will encounter situations where $K \equiv 0$ and J is a domain independent almost complex structure. In these cases, J has to be chosen carefully to achieve regularity, so we will emphasize J and denote the moduli by $\mathcal{M}^J(x_0; x_d, \ldots, x_1)$ therein.

When every element in $\mathcal{M}(x_0; x_d, \ldots, x_1)$ is transversally cut out, $\mathcal{M}(x_0; x_d, \ldots, x_1)$ is a smooth manifold of dimension $|x_0| - \sum_{j=1}^{d} |x_j| + (d-2)$, where $|\cdot|$ denotes the Maslov grading (see Sect. 3.2).

For each transversally cut out rigid element $u \in \mathcal{M}(x_0; x_d, \ldots, x_1)$, we define

$$\begin{aligned} &\mu^u : Hom(E^{d-1}_{x_d(0)}, E^d_{x_d(1)}) \times \cdots \times Hom(E^0_{x_1(0)}, E^1_{x_1(1)}) \to Hom(E^0_{x_0(0)}, E^d_{x_0(1)}) \\ &\mu^u(\psi^d, \ldots, \psi^1)(a) = \operatorname{sign}(u) I_{\partial_d u} \circ \psi^d \circ \cdots \circ \psi^1 \circ I_{\partial_0 u}(a) \end{aligned} \tag{2.10}$$

where $\partial_d u = u|_{\partial_d S}$ for $\partial_d S$ being equipped with the counterclockwise orientation, and $\operatorname{sign}(u) \in \{\pm 1\}$ is the sign determined by u (see "Appendix A"). Finally, we define the A_∞-operation by

$$\begin{aligned} &\mu^d : CF(\mathcal{E}^{d-1}, \mathcal{E}^d) \times \cdots \times CF(\mathcal{E}^0, \mathcal{E}^1) \to CF(\mathcal{E}^0, \mathcal{E}^d) \\ &\mu^d(\psi^d, \ldots, \psi^1) = \sum_{u \in \mathcal{M}(x_0; x_d, \ldots, x_1), u \text{ rigid}} \mu^u(\psi^d, \ldots, \psi^1) \end{aligned} \tag{2.11}$$

The fact that the auxiliary structures can be chosen generically and consistently in the sense of [2], and that $\{\mu^d\}_{d \geqslant 1}$ gives rise to an A_∞ structure follows the argument in [2] line-by-line. This defines the Fukaya category $\mathcal{F}uk(X)$ that we will use throughout.

2.2 Unwinding local systems

The goal of this subsection is to give a computable presentation of $CF(\mathcal{E}^0, \mathcal{E}^1)$, where $\mathcal{E}^i$ are local systems of the same underlying Lagrangian. In particular, the identification (2.16) and (2.27) will be used frequently later.

Let L be a closed exact Lagrangian and $\mathbf{L}$ be its universal cover with covering map $\pi : \mathbf{L} \to L$. Let $o_L \in L$ be a base point of L and we pick a lift $o_{\mathbf{L}} \in \mathbf{L}$ such that $\pi(o_{\mathbf{L}}) = o_L$. We assume throughout that $\Gamma := \pi_1(L, o_L)$ is a finite group so that $\mathbf{L}$ is compact. For each $\mathbf{q} \in \mathbf{L}$, there is a unique path $c_{\mathbf{q}}$ (up to homotopy) from $o_{\mathbf{L}}$ to $\mathbf{q}$ and we identify $\mathbf{q}$ with the homotopy class $[\pi \circ c_{\mathbf{q}}]$. We have a *left* Γ-action on $\mathbf{L}$ given by

$$g\mathbf{q} := g * [(\pi \circ c_{\mathbf{q}})] \tag{2.12}$$

for $g \in \pi_1(L, o_L)$, where $g * [(\pi \circ c_{\mathbf{q}})]$ is a homotopy class of path from o_L to $\pi(\mathbf{q})$ and we identify it as a point in $\mathbf{L}$. It is clear that $h(g\mathbf{q}) = (h * g)\mathbf{q}$. If we pick a Morse function and a Riemannian metric on L to define a Morse cochain complex $C^*(L)$, we can lift the function and metric to $\mathbf{L}$ to define a Morse cochain complex $C^*(\mathbf{L})$. The Γ-action on $\mathbf{L}$ induces a left Γ-action on $C^*(\mathbf{L})$. The Γ-invariant part of $C^*(\mathbf{L})$ can be identified with $C^*(L)$, in other words,

$$C^*(L) = Rhom_{\mathbb{K}[\Gamma]-mod}(\mathbb{K}, C^*(\mathbf{L})) = (C^*(\mathbf{L}))^{\Gamma} \tag{2.13}$$

We want to discuss the analog when L is equipped with local systems.

Given a local system E on L, we use $\mathbf{E} = \pi^* E$ to denote the pull-back local system. For a path $c:[0,1] \to \mathbf{L}$, we use I_c to denote the parallel transport with respect to the pull-back flat connection on $\mathbf{E}$.

Let E^i be local systems on L for $i = 0, 1$. We have *right* actions [see (2.1)]

$$\rho^i : \Gamma \to End(E^i_{o_L}) \tag{2.14}$$

for $i = 0, 1$. It induces a *left* Γ-module structure on $Hom_{\mathbb{K}}(E^0_{o_L}, E^1_{o_L})$ by

$$\psi \mapsto g \cdot \psi := \rho^1(g^{-1}) \circ \psi \circ \rho^0(g) \tag{2.15}$$

Lemma 2.2 *Let E^i be local systems on L for $i = 0, 1$. Then there is a DG left Γ-module isomorphism*

$$\Phi : CF((\mathbf{L}, \mathbf{E}^0), (\mathbf{L}, \mathbf{E}^1)) \simeq C^*(\mathbf{L}) \otimes_{\mathbb{K}} Hom_{\mathbb{K}}(E^0_{o_L}, E^1_{o_L}) \tag{2.16}$$

where the differential on $C^*(\mathbf{L}) \otimes_{\mathbb{K}} Hom_{\mathbb{K}}(E^0_{o_L}, E^1_{o_L})$ is only the differential on the first factor, and the Γ-action on it is given by $g \cdot (x \otimes \psi) := gx \otimes g \cdot \psi$ [see (2.12) and (2.15)].

Proof We use the Morse model to compute the Floer cochain complex. Let $C^*(L)$ be a Morse cochain complex and $C^*(\mathbf{L})$ be its lift. We use ∂_L and $\partial_{\mathbf{L}}$ to denote the differential of $C^*(L)$ and $C^*(\mathbf{L})$, respectively.

For each $\mathbf{q} \in \mathbf{L}$ and both $i = 0, 1$, there is a canonical identification

$$I_{c_{\mathbf{q}}^{-1}} : \mathbf{E}^i_{\mathbf{q}} \to \mathbf{E}^i_{o_{\mathbf{L}}} \tag{2.17}$$

where $c_{\mathbf{q}}$ is the unique (up to homotopy) path from $o_{\mathbf{L}}$ to $\mathbf{q}$. Therefore, it induces a trivialization of $\mathbf{E}^i$. We can also trivialize $Hom_{\mathbb{K}}(\mathbf{E}^0, \mathbf{E}^1)$ using the canonical isomorphism

$$Hom_{\mathbb{K}}(\mathbf{E}^0_{\mathbf{q}}, \mathbf{E}^1_{\mathbf{q}}) \to Hom_{\mathbb{K}}(\mathbf{E}^0_{o_{\mathbf{L}}}, \mathbf{E}^1_{o_{\mathbf{L}}}) = Hom_{\mathbb{K}}(E^0_{o_L}, E^1_{o_L}) \tag{2.18}$$

$$\psi \mapsto I^1_{c_{\mathbf{q}}^{-1}} \circ \psi \circ I^1_{c_{\mathbf{q}}} \tag{2.19}$$

Using the trivialization (2.18), (2.19), we have a graded vector space isomorphism (2.16).

To compare the differential on both sides of (2.16), let $\mathbf{u}$ be a Morse trajectory from $\mathbf{q}_0$ to $\mathbf{q}_1$ contributing to $\partial_{\mathbf{L}}$ and hence the differential of $CF((\mathbf{L},\mathbf{E}^0),(\mathbf{L},\mathbf{E}^1))$.

For $\mathbf{q}_1 \otimes \psi \in C^*(\mathbf{L}) \otimes_{\mathbb{K}} Hom_{\mathbb{K}}(E^0_{o_L}, E^1_{o_L})$,

$$\Phi(\mu^{\mathbf{u}}(\Phi^{-1}(\mathbf{q}_1 \otimes \psi))) \tag{2.20}$$
$$= \mathrm{sign}(\mathbf{u})\mathbf{q}_0 \otimes I_{c^{-1}_{\mathbf{q}_0}} I_{\partial_1 \mathbf{u}} I_{c_{\mathbf{q}_1}} \psi I_{c^{-1}_{\mathbf{q}_1}} I_{\partial_0 \mathbf{u}} I_{c_{\mathbf{q}_0}} \tag{2.21}$$
$$= \mathrm{sign}(\mathbf{u})\mathbf{q}_0 \otimes \psi \tag{2.22}$$

where the second equality uses the fact that $\pi_1(\mathbf{L}) = 1$. Therefore, Φ is an isomorphism of differential graded vector spaces if we define the differential on $C^*(\mathbf{L}) \otimes_{\mathbb{K}} Hom_{\mathbb{K}}(E^0_{o_L}, E^1_{o_L})$ to be $\partial_{\mathbf{L}}$ acting on the first factor.

Finally, we want to compare the left Γ-module structures. In $CF((\mathbf{L},\mathbf{E}^0),(\mathbf{L},\mathbf{E}^1))$, the action on $\psi \in Hom_{\mathbb{K}}(\mathbf{E}^0_{\mathbf{q}}, \mathbf{E}^1_{\mathbf{q}}) = Hom_{\mathbb{K}}(E^0_q, E^1_q)$ is given by

$$\psi \mapsto g\psi = \psi \tag{2.23}$$

where the last ψ lies in $Hom_{\mathbb{K}}(\mathbf{E}^0_{g\mathbf{q}}, \mathbf{E}^1_{g\mathbf{q}}) = Hom_{\mathbb{K}}(E^0_q, E^1_q)$. For $\mathbf{q} \otimes \psi \in C^*(\mathbf{L}) \otimes_{\mathbb{K}} Hom_{\mathbb{K}}(E^0_{o_L}, E^1_{o_L})$,

$$\Phi(g(\Phi^{-1}(\mathbf{q} \otimes \psi))) \tag{2.24}$$
$$= g\mathbf{q} \otimes I_{c^{-1}_{g\mathbf{q}}} I_{c_{\mathbf{q}}} \psi I_{c^{-1}_{\mathbf{q}}} I_{c_{g\mathbf{q}}} \tag{2.25}$$
$$= g\mathbf{q} \otimes I_{g^{-1}} \psi I_g \tag{2.26}$$

which is exactly the one given in (2.12) and (2.15). It finishes the proof. □

We have the following consequence of Lemma 2.2:

Lemma 2.3 *Let E^i be local systems on L for $i = 0, 1$. Then*

$$CF(\mathcal{E}^0, \mathcal{E}^1) = Rhom_{\mathbb{K}[\Gamma]-mod}(\mathbb{K}, C^*(\mathbf{L}) \otimes_{\mathbb{K}} Hom_{\mathbb{K}}(E^0_{o_L}, E^1_{o_L})) \tag{2.27}$$

Proof We use the notation in the proof of Lemma 2.2. Let u be a Morse trajectory from q_0 to q_1 contributing to $\partial_L(q_1)$. Let $\mathbf{q}_0 \in \mathbf{L}$ be a lift of q_0 and let $\mathbf{q}_1 \in \mathbf{L}$ be the corresponding lift of q_1 such that u lifts to a Morse trajectory $\mathbf{u}$ from $\mathbf{q}_0$ to $\mathbf{q}_1$. Let $\psi \in Hom_{\mathbb{K}}(E^0_{q_1}, E^1_{q_1})$. By (2.10), we have

$$\mu^u(\psi) = \mathrm{sign}(u) I_{\partial_1 u} \psi I_{\partial_0 u} \tag{2.28}$$

In the above notation, we regard u as a degenerated holomorphic strip and have suppressed the direction of parallel transport for brevity, since it should be clear from the context.

By definition, $Hom_{\mathbb{K}}(\mathbf{E}^0_{g\mathbf{q}_j}, \mathbf{E}^1_{g\mathbf{q}_j}) \cong Hom_{\mathbb{K}}(E^0_{q_j}, E^1_{q_j})$ for $j = 0, 1$ and for all $g \in \Gamma$. Therefore, for $\psi \in Hom_{\mathbb{K}}(\mathbf{E}^0_{g\mathbf{q}_1}, \mathbf{E}^1_{g\mathbf{q}_1}) \subset CF((\mathbf{L},\mathbf{E}^0),(\mathbf{L},\mathbf{E}^1))$,

$$\mu^{g\mathbf{u}}(\psi) = \mathrm{sign}(g\mathbf{u}) I_{\partial_1 g\mathbf{u}} \psi I_{\partial_0 g\mathbf{u}} \tag{2.29}$$
$$= \mathrm{sign}(u) I_{\partial_1 u} \psi I_{\partial_0 u} \tag{2.30}$$

where $\mu^{g\mathbf{u}}$ is the term in the differential of $CF((\mathbf{L}, \mathbf{E}^0), (\mathbf{L}, \mathbf{E}^1))$ contributed by $g\mathbf{u}$, and the second equality uses $Hom_{\mathbb{K}}(\mathbf{E}^0_{g\mathbf{q}_1}, \mathbf{E}^1_{g\mathbf{q}_1}) \cong Hom_{\mathbb{K}}(E^0_{q_1}, E^1_{q_1})$. The Γ action on the generators ($\mathbf{q} \otimes \psi \mapsto g\mathbf{q} \otimes \psi$) and differentials ($\mu^u \mapsto \mu^{g\mathbf{u}}$) of $CF((\mathbf{L}, \mathbf{E}^0), (\mathbf{L}, \mathbf{E}^1))$ are free and the invariant part can be identified with $CF(\mathcal{E}^0, \mathcal{E}^1)$ so

$$CF(\mathcal{E}^0, \mathcal{E}^1) = Rhom_{\mathbb{K}[\Gamma]-mod}(\mathbb{K}, CF((\mathbf{L}, \mathbf{E}^0), (\mathbf{L}, \mathbf{E}^1))) = (CF((\mathbf{L}, \mathbf{E}^0), (\mathbf{L}, \mathbf{E}^1)))^{\Gamma} \tag{2.31}$$

and the result follows from Lemma 2.2. □

2.3 The universal local system

In this subsection, we introduce the universal local system and hence, in particular, the object $\mathcal{P}$ in Theorem 1.2. Some elementary properties of the universal local system will also be given. Let us start from a general discussion of universal local systems.

Definition 2.4 (*Universal local system*) The **universal local system** E on L is a local system that is uniquely determined by the following conditions: As a vector space, $E_q = \mathbb{K}\langle \pi^{-1}(q) \rangle$ for $q \in L$. For any $y \in \pi^{-1}(q)$ and $c : [0, 1] \to L$ such that $c(0) = q$, the parallel transport of E satisfies $I_c(y) = \mathbf{c}(1)$, where $\mathbf{c} : [0, 1] \to \mathbf{L}$ is the unique path such that $\pi \circ \mathbf{c} = c$ and $\mathbf{c}(0) = y$.

As usual, we have the monodromy right Γ-action ρ on E_{o_L} (2.1). On top of that, we can use the left Γ action on $\mathbf{L}$ (2.12) to induce (by extending it linearly) a left Γ action on E_q for all $q \in L$. These two actions on E_{o_L} commute and in general, we have

Lemma 2.5 *Let E be the universal local system on L. For $q \in L$, $y \in E_q$, $g \in \Gamma$ and $c : [0, 1] \to L$ such that $c(0) = q$, we have*

$$g(I_c y) = I_c(gy) \tag{2.32}$$

Proof Without loss of generality, let $y \in \pi^{-1}(q)$. We can identify y with the homotopy class $[\pi \circ c_y]$ from o_L to q. Then we have [see (2.12)]

$$g(I_c y) = g * [\pi \circ c_y] * [c] = I_c(gy) \tag{2.33}$$

where $[c]$ is the homotopy class of path from $c(0)$ to $c(1)$ that c represents. □

Let $\mathcal{E} = (L, E, \nabla)$. Since we have a left action on E_q for all $q \in L$, it induces a left Γ action on $CF(\mathcal{E}', \mathcal{E})$

$$\psi \mapsto g\psi \tag{2.34}$$

for any $\mathcal{E}' \in Ob(\mathcal{F})$. Similarly, for any $\mathcal{E}' \in Ob(\mathcal{F})$, we have the induced right Γ action on $CF(\mathcal{E}, \mathcal{E}')$

$$\psi(\cdot) \mapsto \psi(g\cdot) \tag{2.35}$$

As an immediate consequence of Lemma 2.5 and the definition of μ^u [see (2.10)], we have

Corollary 2.6 *Let E be the universal local system on L. Let $\mathcal{L}_1 \dots, \mathcal{L}_r, \mathcal{K}_1, \dots, \mathcal{K}_s \in Ob(\mathcal{F})$. Let $y_j \in CF(\mathcal{K}_j, \mathcal{K}_{j+1})$ for $j = 1, \dots, s-1$, $x_j \in CF(\mathcal{L}_j, \mathcal{L}_{j+1})$ for $j = 1, \dots, r-1$, $\psi_2 \in CF(\mathcal{E}, \mathcal{K}_1)$ and $\psi_1 \in CF(\mathcal{L}_r, \mathcal{E})$, we have*

$$\mu^u(y_{s-1}, \dots, y_1, \psi_2 g, \psi_1, x_{r-1}, \dots, x_1) = \mu^u(y_{s-1}, \dots, y_1, \psi_2, g\psi_1, x_{r-1}, \dots, x_1) \tag{2.36}$$

for all $g \in \Gamma$, where u is an element in the appropriate moduli contributing to the A_∞-structural maps of $\mathcal{F}$. When $s = 0$ (resp. $r = 0$), we have

$$g\mu^u(\psi_1, x_{r-1}, \dots, x_1) = \mu^u(g\psi_1, x_{r-1}, \dots, x_1), \text{ respectively} \tag{2.37}$$

$$\mu^u(y_{s-1}, \dots, y_1, \psi_2 g) = \mu^u(y_{s-1}, \dots, y_1, \psi_2)g \tag{2.38}$$

Remark 2.7 We offer an alternative way to understand (2.36) using $\mathbf{L}$ instead of $\mathcal{E}$. For each $q \in L$ and a lift $\mathbf{q}$ of q, we can view $\mathbf{q}$ as a point in $\mathbf{L}$ or as an element in E_q. Therefore, we can identify the generators of $CF(T_q^*L, \mathcal{E})$ and the generators of $CF(\bigcup_{g\in\Gamma} T_{g\mathbf{q}}^*\mathbf{L}, \mathbf{L})$ by

$$E_q \ni \mathbf{q} \mapsto \mathbf{q} \in T_{\mathbf{q}}^*\mathbf{L} \cap \mathbf{L} \tag{2.39}$$

Dually, $CF(\mathcal{E}, T_q^*L)$ can be identified with $CF(\mathbf{L}, \cup_{g\in\Gamma} T_{g\mathbf{q}}^*\mathbf{L})$ by

$$Hom_{\mathbb{K}}(E_q, \mathbb{K}) \ni \mathbf{q}^\vee \mapsto \mathbf{q} \in \mathbf{L} \cap T_{\mathbf{q}}^*\mathbf{L} \tag{2.40}$$

The right action (2.35) on $Hom_{\mathbb{K}}(E_q, \mathbb{K})$ is given by $\mathbf{q}^\vee g = (g^{-1}\mathbf{q})^\vee$, which corresponds to the right action on $\mathbf{L}$ by $\mathbf{q}g = g^{-1}\mathbf{q}$.

Now, we want to make connection with Corollary 2.6.

For simplicity, we assume that K_1 and L_1 are Lagrangians without local systems and $\psi_1 = \mathbf{q}_1 \in E_{q_1}$, $\psi_2 = \mathbf{q}_2^\vee \in Hom_{\mathbb{K}}(E_{q_2}, \mathbb{K})$. Let γ be $\partial_{r+1}S$, which is the component of ∂S with label L.

Since the parallel transport of E can be identified with moving the points in $\mathbf{L}$, for μ^u to be non-zero and contribute to the RHS of (2.36), there is exactly one $g \in \Gamma$ and one lift of $u|_\gamma$, which is denoted by $\mathbf{u} : \gamma \to \mathbf{L}$, such that $\mathbf{u}$ goes from $g\mathbf{q}_1$ to $\mathbf{q}_2$. For each $h \in \Gamma$, the maps $h\mathbf{u} : \gamma \to \mathbf{L}$ are the other lifts of $u|_\gamma$ and $h\mathbf{u}$ goes from $hg\mathbf{q}_1$ to $h\mathbf{q}_2$.

Roughly speaking, one can define a Floer theory by counting $(u, \mathbf{u})$, where u is as in Corollary 2.6 and $\mathbf{u}$ is a lift of $u|_\gamma$. This definition is explained in details in [30] and the outcome is the same as Lagrangian Floer theory with local systems. In this setting, the pair $(u, h\mathbf{u})$ contributes to

$$\mu^{(u,h\mathbf{u})}(y_{s-1},\ldots,y_1,h\mathbf{q}_2,hg\mathbf{q}_1,x_{r-1},\ldots,x_1) \tag{2.41}$$

and it equals to $\mu^{(u,\mathbf{u})}(y_{s-1},\ldots,y_1,\mathbf{q}_2,g\mathbf{q}_1,x_{r-1},\ldots,x_1)$. Under the identification (2.39), (2.40), it means that (when $h = g^{-1}$)

$$\begin{aligned}&\mu^{(u,\mathbf{u})}(y_{s-1},\ldots,y_1,\mathbf{q}_2^\vee,g\mathbf{q}_1,x_{r-1},\ldots,x_1)\\&\quad=\mu^{(u,g^{-1}\mathbf{u})}(y_{s-1},\ldots,y_1,(g^{-1}\mathbf{q}_2)^\vee,\mathbf{q}_1,x_{r-1},\ldots,x_1)\end{aligned}$$

which is exactly the same as (2.36)

The rest of this subsection is devoted to the self-Floer chain complex $CF(\mathcal{E},\mathcal{E})$ when E is the universal local system of L. Let $R := \mathbb{K}[\Gamma]$ and 1_Γ be the unit of Γ. For $h \in \Gamma$, we define $\tau_h \in Hom_{\mathbb{K}}(R, R)$ by

$$\tau_h(g) = \begin{cases} 1_\Gamma & \text{if } g = h^{-1} \\ 0 & \text{if } g \in \Gamma\backslash\{h^{-1}\}\end{cases} \tag{2.42}$$

Note that $R \cong E_{o_L}$ so, by Lemma 2.3, we have

$$CF(\mathcal{E},\mathcal{E}) = (C^*(\mathbf{L}) \otimes Hom_{\mathbb{K}}(R, R))^\Gamma \tag{2.43}$$

as a Γ-module.

In particular, we have μ^1, μ^2 on $(C^*(\mathbf{L}) \otimes Hom_{\mathbb{K}}(R, R))^\Gamma$ inherited from $CF(\mathcal{E},\mathcal{E})$. In Lemma 2.2, we proved that μ^1 coincides with the Morse differential $\partial_{\mathbf{L}}$ on the first factor. The same line of argument can prove that μ^2 coincides with the Floer multiplication on $C^*(\mathbf{L})$ tensored with the composition in $Hom_{\mathbb{K}}(R, R)$ (i.e. $\mu^2_{\mathbf{L}}(-,-) \otimes - \circ -$).

Let $\Phi_2 : C^*(\mathbf{L}) \otimes R \to (C^*(\mathbf{L}) \otimes Hom_{\mathbb{K}}(R, R))^\Gamma$ be the graded vector space isomorphism given by

$$\Phi_2 : x \otimes h \mapsto \sum_{g\in\Gamma} gx \otimes g \cdot \tau_h = \sum_{g\in\Gamma} gx \otimes I_{g^{-1}}\tau_h I_g \tag{2.44}$$

Lemma 2.8 *We have the following equalities*

$$\Phi_2^{-1} \circ \mu^1 \circ \Phi_2(x \otimes h) = \partial_{\mathbf{L}}(x) \otimes h \tag{2.45}$$

$$\Phi_2^{-1} \circ \mu^2 \circ (\Phi_2(x_2 \otimes h_2), \Phi_2(x_1 \otimes h_1)) = \mu^2_{\mathbf{L}}(x_2, h_2x_1) \otimes h_2h_1 \tag{2.46}$$

As a consequence of (2.45)*, we have* $H^*(CF(\mathcal{E},\mathcal{E})) = H^*(\mathbf{L}) \otimes R$ *as a vector space.*

Proof For $x \otimes h \in C^*(\mathbf{L}) \otimes R$,

$$\Phi_2^{-1} \circ \mu^1 \circ \Phi_2(x \otimes h) \tag{2.47}$$

$$= \Phi_2^{-1}\left(\sum_{g\in\Gamma} \partial_{\mathbf{L}}(gx) \otimes I_{g^{-1}}\tau_h I_g\right) \tag{2.48}$$

$$= \Phi_2^{-1}\left(\sum_{g\in\Gamma} g\partial_{\mathbf{L}}(x) \otimes I_{g^{-1}}\tau_h I_g\right) \tag{2.49}$$

$$= \partial_{\mathbf{L}}(x) \otimes h \tag{2.50}$$

where the second equality uses Corollary 2.6.

For $x_i \otimes h_i \in C^*(\mathbf{L}) \otimes R$, $i = 1, 2$, we have

$$\Phi_2^{-1} \circ \mu^2 \circ (\Phi_2(x_2 \otimes h_2), \Phi_2(x_1 \otimes h_1)) \tag{2.51}$$

$$= \Phi_2^{-1}\left(\sum_{g_1,g_2\in\Gamma} \mu^2_{\mathbf{L}}(g_2x_2, g_1x_1) \otimes I_{g_2^{-1}}\tau_{h_2} I_{g_2} I_{g_1^{-1}}\tau_{h_1} I_{g_1}\right) \tag{2.52}$$

For $\tau_{h_2} I_{g_2} I_{g_1^{-1}}\tau_{h_1}$ and hence $I_{g_2^{-1}}\tau_{h_2} I_{g_2} I_{g_1^{-1}}\tau_{h_1} I_{g_1}$ to be non-zero, we must have

$$1_\Gamma * g_1^{-1} * g_2 = h_2^{-1} \tag{2.53}$$

and for any g_2, there is a unique g_1 $(= g_2h_2)$ such that $g_1^{-1}g_2 = h_2^{-1}$. Therefore, the sum becomes

$$\Phi_2^{-1}\left(\sum_{g_2\in\Gamma} \mu^2_{\mathbf{L}}(g_2x_2, g_2h_2x_1) \otimes I_{g_2^{-1}}\tau_{h_2} I_{h_2^{-1}}\tau_{h_1} I_{g_1} I_{g_2^{-1}} I_{g_2}\right) \tag{2.54}$$

$$= \Phi_2^{-1}\left(\sum_{g_2\in\Gamma} g_2\mu^2_{\mathbf{L}}(x_2, h_2x_1) \otimes I_{g_2^{-1}}\tau_{h_2} I_{h_2^{-1}}\tau_{h_1} I_{h_2} I_{g_2}\right) \tag{2.55}$$

$$= \Phi_2^{-1}\left(\sum_{g_2\in\Gamma} g_2\mu^2_{\mathbf{L}}(x_2, h_2x_1) \otimes I_{g_2^{-1}}\tau_{h_2h_1} I_{g_2}\right) \tag{2.56}$$

$$= \mu^2_{\mathbf{L}}(x_2, h_2x_1) \otimes h_2h_1 \tag{2.57}$$

where the second equality uses that $\mu^2_{\mathbf{L}}$ is Γ-equivariant, and the third equality uses $\tau_{h_2} I_{h_2^{-1}}\tau_{h_1} I_{h_2} = \tau_{h_1} I_{h_2} = \tau_{h_2h_1}$. □

2.4 Spherical Lagrangians

In this subsection, we apply the results from the previous subsections to the case that $L = P$ such that

$$P \text{ is diffeomorphic to } S^n/\Gamma \text{ for some } \Gamma \subset SO(n+1),$$
$$\text{so that the } \Gamma\text{-action is free, and } P \text{ is spin} \tag{2.58}$$

Remark 2.9 A finite free quotient of a sphere S^n/Γ is spin if and only if there exists $\widetilde{\Gamma} \subset Spin(n+1)$ such that the covering homomorphism $Spin(n+1) \to SO(n+1)$ restricts to an isomorphism $\widetilde{\Gamma} \simeq \Gamma$.

First, we apply the discussion from Sect. 2.2.

Lemma 2.10 *Let E^i be local systems on P for $i = 0, 1$. If* char$(\mathbb{K})$ *does not divide* $|\Gamma|$*, then $HF(\mathcal{E}^0, \mathcal{E}^1) = H^*(S^n) \otimes Hom_{\mathbb{K}[\Gamma]}(\mathcal{E}^0_{o_L}, \mathcal{E}^1_{o_L})$ as a $\mathbb{K}$-vector space.*

Proof We apply the Leray spectral sequence to Lemma 2.3. The E_2-page is given by

$$E_2^{p,q} = H^p(\Gamma, H^q(S^n) \otimes_{\mathbb{K}} Hom_{\mathbb{K}}(\mathcal{E}^0_{o_L}, \mathcal{E}^1_{o_L}))$$

where the Γ-action is given by $x \otimes \psi \mapsto x \otimes g \cdot \psi$ and $g \cdot \psi = \rho^1(g^{-1}) \circ \psi \circ \rho^0(g)$. As a result, we have

$$E_2^{p,q} = H^q(S^n) \otimes Ext^p_{\Gamma}(\Gamma, Hom_{\mathbb{K}}(\mathcal{E}^0_{o_L}, \mathcal{E}^1_{o_L}))$$

When char$(\mathbb{K})$ does not divide $|\Gamma|$, $\mathbb{K}[\Gamma]$ is semi-simple by Maschke's theorem. Therefore, $Ext^p_{\Gamma}(\Gamma, Hom_{\mathbb{K}}(\mathcal{E}^0_{o_L}, \mathcal{E}^1_{o_L})) \neq 0$ only if $p = 0$. It implies that the spectral sequence degenerate at E_2-page and the result follows from the fact that $Ext^0_{\Gamma}(\Gamma, Hom_{\mathbb{K}}(\mathcal{E}^0_{o_L}, \mathcal{E}^1_{o_L}))$ consists of $\psi \in Hom_{\mathbb{K}}(\mathcal{E}^0_{o_L}, \mathcal{E}^1_{o_L})$ such that $g \cdot \psi = \psi$, which is clearly $Hom_{\mathbb{K}[\Gamma]}(\mathcal{E}^0_{o_L}, \mathcal{E}^1_{o_L})$. □

Corollary 2.11 *Let $\mathcal{E}^0$ be any local system on P corresponding to an irreducible representation of* Γ*. If* char$(\mathbb{K})$ *does not divide* $|\Gamma|$*, then $HF(\mathcal{E}^0, \mathcal{E}^0) = H^*(S^n)$.*

Proof It follow from Lemma 2.10 and Schur's lemma $Hom_{\mathbb{K}[\Gamma]}(\mathcal{E}^0_{o_L}, \mathcal{E}^0_{o_L}) = \mathbb{K}$.

Notice that, the ring structure is also determined uniquely by dimension and degree reason. □

Now, we want to compute the cohomological endomorphism algebra structure of the universal local system on P using Lemma 2.8. Since the universal local system on P plays a distinguished role in the paper, we denote it by $\mathcal{P}$. We define μ^1, μ^2 on $C^*(\mathbf{P}) \otimes R$ by (2.45) and (2.46), respectively. By (2.45), we know that $H^*(C^*(\mathbf{P}) \otimes R)$ is given by $H^*(\mathbf{P}) \otimes R$. We are going to determine the algebra structure in the next lemma. Before that, we recall a convention

Convention 2.12 *If C is a differential graded algebra (e.g. a $\mathbb{K}$-algebra with no differential), then C is viewed as an A_∞ algebra by*

$$\mu^1(a) = (-1)^{|a|}\partial(a) \tag{2.59}$$
$$\mu^2(a_1, a_0) = (-1)^{|a_0|} a_1 a_0 \tag{2.60}$$

and $\mu^k = 0$ for $k \geqslant 3$, where $a, a_0, a_1 \in C$ and ∂ is the differential of C.

Lemma 2.13 *Let $\mathcal{P}$ be the universal local system on P and $R := \mathbb{K}[\Gamma]$. Then the Floer cohomology $HF(\mathcal{P}, \mathcal{P}) = H^*(S^n) \otimes_{\mathbb{K}} R$ as a $\mathbb{K}$-algebra, where the ring structure on the right is the product of the standard ring structure.*

Proof Pick a Morse model such that $C^*(P)$ has only one degree 0 generator e and one degree n generator f. The corresponding Morse complex $C^*(\mathbf{P})$ has $|\Gamma|$ degree 0 generator $\{g\mathbf{e}\}_{g\in\Gamma}$ and $|\Gamma|$ degree n generator $\{g\mathbf{f}\}_{g\in\Gamma}$. It is clear that $\sum_g g\mathbf{e}$ represents the unit of $H^0(\mathbf{P})$. Therefore, $\{[\sum_g g\mathbf{e}] \otimes h\}_{h\in\Gamma}$ are the degree 0 generators of $H(C^*(\mathbf{P}) \otimes R)$ [see the correspondence of (2.44) (2.45)].

Similarly, if x represents a generator of $H^n(\mathbf{P})$, then $\{[x] \otimes h\}_{h\in\Gamma}$ are the degree n generators of $H(C^*(\mathbf{P}) \otimes R)$. It follows from (2.46) that

$$\mu^2\left(\left[\sum_g g\mathbf{e}\right] \otimes h_2, \left[\sum_g g\mathbf{e}\right] \otimes h_1\right) = \left[\sum_g g\mathbf{e}\right] \otimes h_2h_1 \tag{2.61}$$

$$\mu^2\left([x] \otimes h_2, \left[\sum_g g\mathbf{e}\right] \otimes h_1\right) = [x] \otimes h_2h_1 \tag{2.62}$$

$$\mu^2\left(\left[\sum_g g\mathbf{e}\right] \otimes h_2, [x] \otimes h_1\right) = (-1)^{|x|}[h_2x] \otimes h_2h_1 = (-1)^{|x|}[x] \otimes h_2h_1 \tag{2.63}$$

Therefore, $H(C^*(\mathbf{P}) \otimes R) = H^*(S^n) \otimes_{\mathbb{K}} R$ as a $\mathbb{K}$-algebra (see Convention 2.12). The result now follows from Lemma 2.3, 2.8 [see (2.31), (2.44)]. $\square$

Let $\theta_g = 1_{H^0(S^n)} \otimes g \in H^0(S^n) \otimes R$. By Lemma 2.13, we have a left Γ-action on $HF(\mathcal{E}, \mathcal{P})$ given by

$$x \mapsto [\mu^2(\theta_g, x)] \tag{2.64}$$

for any $\mathcal{E} \in \mathcal{F}$. On the other hand, we have another left Γ-action on $CF(\mathcal{E}, \mathcal{P})$ given by (2.34), which descends to a left Γ-action on the cohomology $HF(\mathcal{E}, \mathcal{P})$.

Lemma 2.14 *When $\mathcal{E} = \mathcal{P}$, the two left Γ-actions (2.64) and (2.34) on $\theta_{1_\Gamma} \in HF(\mathcal{P}, \mathcal{P})$ coincide.*

Proof We use the notations in the proof of Lemma 2.13. The element $\theta_h \in H^0(S^n) \otimes R$ is represented by $\sum_g g\mathbf{e} \otimes h \in C^0(\mathbf{P}) \otimes R$. We have [see (2.44)]

$$\Phi_2\left(\sum_g g\mathbf{e} \otimes h\right) = \sum_{g_2, g_1} g_2g_1\mathbf{e} \otimes I_{g_2^{-1}}\tau_h I_{g_2} \tag{2.65}$$

$$= \sum_{g} g\mathbf{e} \otimes I_{g^{-1}} \left(\sum_{g'} I_{g'} \tau_h I_{(g')^{-1}} \right) I_g \tag{2.66}$$

Undoing the trivialization (2.16), we have

$$\Phi^{-1}\left(\Phi_2\left(\sum_{g} g\mathbf{e} \otimes h\right)\right) = \sum_{g} g\mathbf{e} \otimes I_{c_\mathbf{e}} \left(\sum_{g'} I_{g'} \tau_h I_{(g')^{-1}} \right) I_{c_\mathbf{e}^{-1}} \tag{2.67}$$

where $c_\mathbf{e} : [0, 1] \to \mathbf{P}$ is a path from $o_\mathbf{P}$ to $\mathbf{e}$. With respect to the identification $(CF((\mathbf{P}, \mathbf{E}), (\mathbf{P}, \mathbf{E})))^\Gamma = CF(\mathcal{P}, \mathcal{P})$ [see (2.31)],

$$\begin{aligned} &\Phi^{-1}\left(\Phi_2\left(\sum_{g} g\mathbf{e} \otimes h\right)\right) \\ &\quad = I_{\pi \circ c_\mathbf{e}} \left(\sum_{g'} I_{g'} \tau_h I_{(g')^{-1}} \right) I_{(\pi \circ c_\mathbf{e})^{-1}} \in Hom(E_e, E_e) \subset CF(\mathcal{P}, \mathcal{P}) \end{aligned} \tag{2.68}$$

Without loss of generality, we can assume $e = o_L$ so

$$\sum_{g'} I_{g'} \tau_h I_{(g')^{-1}} \in Hom(E_{o_L}, E_{o_L}) \subset CF(\mathcal{P}, \mathcal{P}) \tag{2.69}$$

represents θ_h under the isomorphism $HF^0(\mathcal{P}, \mathcal{P}) = H^0(S^n) \otimes R$.

For each $y \in \Gamma \subset E_{o_L}$, there is a unique $g'(= h * y)$ such that $\tau_h I_{(g')^{-1}}(y) \neq 0$. Therefore, $\sum_{g'} I_{g'} \tau_h I_{(g')^{-1}}(y) = hy$ for all $y \in E_{o_L}$. In particular, it means that

$$\sum_{g'} I_{g'} \tau_h I_{(g')^{-1}} = h \left(\sum_{g'} I_{g'} \tau_{1_\Gamma} I_{(g')^{-1}} \right) \tag{2.70}$$

so $\theta_h = h\theta_{1_\Gamma}$ and hence $\mu^2(\theta_h, \theta_{1_\Gamma}) = (-1)^{|\theta_{1_\Gamma}|}\theta_h = h\theta_{1_\Gamma}$ as desired. □

Remark 2.15 From the proof of Lemma 2.14, we see that the identity morphism at E_{o_L} represents the cohomological unit. It is in general true that if one picks a Morse cochain complex for a Lagrangian submanifold L such that there is a unique degree 0 generator e_L representing the cohomological unit of $C^*(L)$, then the identity morphism of E_{o_L} is a cohomological unit of $CF(\mathcal{E}, \mathcal{E})$, where $\mathcal{E}$ is a local system on L.

Corollary 2.16 *The two left Γ-actions (2.64) and (2.34) on $HF^k(\mathcal{E}, \mathcal{P})$ coincide, up to $(-1)^k$, for all $\mathcal{E} \in \mathcal{F}$.*

Proof Let $x \in HF(\mathcal{E}, \mathcal{P})$. We have

$$[\mu^2(\theta_g, x)] = [\mu^2(\mu^2(\theta_g, \theta_{1_\Gamma}), x)] \tag{2.71}$$
$$= [\mu^2(g\theta_{1_\Gamma}, x)] \tag{2.72}$$
$$= [g\mu^2(\theta_{1_\Gamma}, x)] \tag{2.73}$$
$$= (-1)^{|x|} gx \tag{2.74}$$

where the first equality uses Lemma 2.13, the second equality uses Lemma 2.14, the third equality uses Corollary 2.6 and the last equality uses that θ_{1_Γ} is a cohomological unit. ◻

Similarly, for any $\mathcal{E} \in \mathcal{F}$, we have a right Γ-action on $HF(\mathcal{P}, \mathcal{E})$ given by

$$x \mapsto [\mu^2(x, \theta_g)] \tag{2.75}$$

and another right action on $HF(\mathcal{P}, \mathcal{E})$ given by (2.35). The analog of Corollary 2.16 holds, (i.e. $\mu^2(\theta_{1_\Gamma}, \theta_h) = \theta_h = \theta_{1_\Gamma} h$) and we leave the details to readers.

Corollary 2.17 *The two right Γ-actions (2.75) and (2.35) on $HF(\mathcal{P}, \mathcal{E})$ coincide (without additional factor of -1) for all $\mathcal{E} \in \mathcal{F}$.*

2.5 Equivariant evaluation

In this subsection, we want to give the definition of

$$T_{\mathcal{P}}(\mathcal{E}) := Cone(hom_{\mathcal{F}}(\mathcal{P}, \mathcal{E}) \otimes_\Gamma \mathcal{P} \xrightarrow{ev} \mathcal{E}) \tag{2.76}$$

that arises in (1.1) in the context of Fukaya category. We will keep the exposition minimal and self-contained here.

Let $\mathcal{F}^{\mathrm{perf}}$ be the DG category of perfect A_∞ right modules over $\mathcal{F}$. We have a cohomologically full and faithful Yoneda embedding [2, Section (2g)]

$$\mathcal{Y} : \mathcal{F} \to \mathcal{F}^{\mathrm{perf}} \tag{2.77}$$

By abuse of notation, we use $\mathcal{E}$ to denote $\mathcal{Y}(\mathcal{E})$ for $\mathcal{E} \in Ob(\mathcal{F})$.

Let P be a Lagrangian brane such that $\pi_1(P) = \Gamma$, and $\mathcal{P}$ be the object with underlying Lagrangian P equipped with the universal local system E. Let $\mathcal{E} \in Ob(\mathcal{F})$. By Corollary 2.6, we know that [see (2.35)]

$$\mu^1_{\mathcal{F}}(\psi)g = \mu^1_{\mathcal{F}}(\psi g) \tag{2.78}$$

for $\psi \in hom_{\mathcal{F}}(\mathcal{P}, \mathcal{E})$ so $hom_{\mathcal{F}}(\mathcal{P}, \mathcal{E})$ is a DG right Γ-module.

Given a DG right Γ-module V, we define an object $V \otimes_\Gamma \mathcal{P} \in Ob(\mathcal{F}^{perf})$ as follows: For every $X \in Ob(\mathcal{F})$, we have a cochain complex

$$(V \otimes_\Gamma \mathcal{P})(X) := V \otimes_\Gamma hom_{\mathcal{F}}(X, \mathcal{P}) \tag{2.79}$$

where the left Γ-actions on $hom_{\mathcal{F}}(X, \mathcal{P})$ is given by (2.34). By Corollary 2.6, we have

$$\begin{cases} \mu^1_V(vg) \otimes \psi = \mu^1_V(v)g \otimes \psi \\ v \otimes \mu^1_{\mathcal{F}}(g\psi) = v \otimes g\mu^1_{\mathcal{F}}(\psi) \end{cases} \tag{2.80}$$

for $v \otimes \psi \in V \otimes_\Gamma hom_{\mathcal{F}}(X, \mathcal{P})$ so

$$\mu^{1|0} : v \otimes \psi \mapsto (-1)^{|\psi|-1}\mu^1_V(v) \otimes \psi + v \otimes \mu^1_{\mathcal{F}}(\psi) \tag{2.81}$$

is a well-defined differential on $V \otimes_\Gamma hom_{\mathcal{F}}(X, \mathcal{P})$.

The A_∞ right $\mathcal{F}$ module structure on $V \otimes_\Gamma \mathcal{P}$ is given by

$$\mu^{1|d-1} : (v \otimes \psi, x_{d-1}, \dots, x_1) \mapsto v \otimes \mu^d_{\mathcal{F}}(\psi, x_{d-1}, \dots, x_1) \tag{2.82}$$

for $v \otimes \psi \in V \otimes_\Gamma hom_{\mathcal{F}}(X_d, \mathcal{P})$ and $x_j \in hom_{\mathcal{F}}(X_j, X_{j+1})$. The morphism $\mu^{1|d-1}$ is well-defined by Corollary 2.6 and we leave it to readers to check that $\{\mu^{1|j}\}_{j=0}^\infty$ satisfies A_∞ module relations [2, Equation (1.19)]. In particular, we have an A_∞ right $\mathcal{F}$ module $hom_{\mathcal{F}}(\mathcal{P}, \mathcal{E}) \otimes_\Gamma \mathcal{P}$.

Now we want to define an A_∞ morphism

$$ev_\Gamma : hom_{\mathcal{F}}(\mathcal{P}, \mathcal{E}) \otimes_\Gamma \mathcal{P} \to \mathcal{E} \tag{2.83}$$

as follows. For $\psi^2 \otimes \psi^1 \in hom_{\mathcal{F}}(\mathcal{P}, \mathcal{E}) \otimes_\Gamma hom_{\mathcal{F}}(X_d, \mathcal{P})$ and $x_j \in hom_{\mathcal{F}}(X_j, X_{j+1})$, we define

$$ev^d_\Gamma : (\psi^2 \otimes \psi^1, x_{d-1}, \dots, x_1) \mapsto \mu^{d+1}_{\mathcal{F}}(\psi^2, \psi^1, x_{d-1}, \dots, x_1) \tag{2.84}$$

The well-definedness follows from Corollary 2.6 again. The fact that $ev_\Gamma = \{ev^d_\Gamma\}_{d=1}$ defines an A_∞ morphism follows from the A_∞ relations of $\mathcal{F}$. As a consequence, we can define

$$T_{\mathcal{P}}(\mathcal{E}) := Cone(hom_{\mathcal{F}}(\mathcal{P}, \mathcal{E}) \otimes_\Gamma \mathcal{P} \xrightarrow{ev_\Gamma} \mathcal{E}) \tag{2.85}$$

as the A_∞ mapping cone for the A_∞ morphism ev_Γ (see [2, Section (3e)]). In particular, for $X \in Ob(\mathcal{F})$, we have a cochain complex

$$T_{\mathcal{P}}(\mathcal{E})(X) = (hom_{\mathcal{F}}(\mathcal{P}, \mathcal{E}) \otimes_\Gamma hom_{\mathcal{F}}(X, \mathcal{P}))[1] \oplus hom_{\mathcal{F}}(X, \mathcal{E}) \tag{2.86}$$

with differential and multiplication given by

$$\begin{aligned} \mu^1_{T_{\mathcal{P}}(\mathcal{E})}(\psi^2 \otimes \psi^1, x) = ((-1)^{|\psi^1|-1}\mu^1_{\mathcal{F}}(\psi^2) \otimes \psi^1 \\ + \psi^2 \otimes \mu^1_{\mathcal{F}}(\psi^1), \mu^1_{\mathcal{F}}(x) + \mu^2_{\mathcal{F}}(\psi^2, \psi^1)) \end{aligned} \tag{2.87}$$

$$\mu^2_{T_{\mathcal{P}}(\mathcal{E})}((\psi^2 \otimes \psi^1, x), a) = (\psi^2 \otimes \mu^2_{\mathcal{F}}(\psi^1, a), \mu^2_{\mathcal{F}}(x, a) + \mu^3_{\mathcal{F}}(\psi^2, \psi^1, a)) \tag{2.88}$$

Finally, we want to state a functorial property of $T_{\mathcal{P}}(\mathcal{E})$.

Corollary 2.18 *Let $\mathcal{F}_0$, $\mathcal{F}_1$ be the Fukaya categories with respect to two different sets of choices of auxiliary data. The Lagrangian branes $\mathcal{P}$, $\mathcal{E}$ above will be denoted by $\mathcal{P}_j$, $\mathcal{E}_j$, respectively, when we regard them as objects in $\mathcal{F}_j$, for $j = 0, 1$. Let $\mathcal{G} : \mathcal{F}_0 \to \mathcal{F}_1$ be a quasi-equivalence sending $\mathcal{P}_0$ to $\mathcal{P}_1$ and $\mathcal{E}_0$ to $\mathcal{E}_1$. Then*

$$\mathcal{G}(T_{\mathcal{P}_0}(\mathcal{E}_0)) \simeq T_{\mathcal{P}_1}(\mathcal{E}_1)$$

The proof is straightforward along the same line as [2, Lemma 5.6] and is left to interested readers.

Remark 2.19 A thorough discussion of the categorical notions can be found in [31], which is the extended version of the current paper. The readers can also find an intrinsic proof of Corollary 2.18, and an explanation of Remark 1.4, in [31].

3 Symplectic field theory package

The main goal of this section is to derived the regularity results (Propositions 3.27, 3.29 and 3.30) we need for the later sections. The main ingredient is a trick given in [25], combined with many special features of our setup. For clarity, we recall and specialize some generalities from symplectic field theory to our context, introducing notations that will be used specifically in our proof. This consists the main contents from Sects. 3.1 to 3.5.

The regularity results in this section allow us to establish Proposition 3.32 in Sect. 3.7, which gives us enough control on the bubbling of the moduli of maps we need in Sects. 4 and 5.

For more general backgrounds in symplectic field theory, readers are referred to [24,25,27,32,33] etc.

3.1 The set up

Let (Y, α) be a contact manifold with a contact form α.

Definition 3.1 A cylindrical almost complex structure on the symplectization $SY := (\mathbb{R} \times Y, d(e^r\alpha))$ is an almost complex structure such that

- J is invariant under $\mathbb{R}$ action
- $J(\partial_r) = R_\alpha$, where R_α is the Reeb vector field of α
- $J(\ker(\alpha)) = \ker(\alpha)$
- $d\alpha(\cdot, J\cdot)|_{\ker(\alpha)}$ is a metric on $\ker(\alpha)$

The set of cylindrical almost complex structures is denoted by $\mathcal{J}^{cyl}(Y, \alpha)$. If $I \subset \mathbb{R}$ is an interval, we call J a cylindrical almost complex structure on $(I \times Y, d(e^r\alpha))$ if $J = J'|_{I\times Y}$ for some $J' \in \mathcal{J}^{cyl}(Y, \alpha)$. Let (M, ω, θ) be a Liouville domain with a separating contact hypersurface $(Y, \alpha = \theta|_Y)$ such that $Y \cap \partial M = \varnothing$. By the neighborhood theorem, there is a neighborhood $N(Y) \subset M$ of Y such that we have a symplectomorphism

$$\Phi_{N(Y)} : (N(Y), \omega|_{N(Y)}) \simeq ((-\epsilon, \epsilon) \times Y, d(e^t\alpha)) \tag{3.1}$$

for some $\epsilon > 0$.

Let J^0 be a compatible almost complex structure on M such that $(\Phi_{N(Y)})_*(J^0|_{N(Y)})$ is cylindrical. We say that a smooth family of compatible almost complex structure $(J^\tau)_{\tau\in[0,\infty)}$ on M is *adjusted to* $N(Y)$ if

$$\begin{cases} J^\tau|_{M\backslash N(Y)} = J^0|_{M\backslash N(Y)} \text{ for all } \tau \\ \text{for each } \tau\text{, we have } \Phi^\tau_{N(Y)} : (N(Y), J^\tau|_{N(Y)}) \simeq ((-(\tau+\epsilon), \tau+\epsilon) \times Y, (J^\tau)') \end{cases} \tag{3.2}$$

where $\Phi^\tau_{N(Y)}$ is an isomorphism of almost complex manifolds, the diffeomorphism $\Phi^\tau_{N(Y)} \circ (\Phi_{N(Y)})^{-1}$ is the identity on the Y factor, and $(J^\tau)'$ is the unique cylindrical almost complex structure such that $(J^\tau)'|_{(-\epsilon,\epsilon)\times Y} = (\Phi_{N(Y)})_*(J^0|_{N(Y)})$.

Let M^- be the Liouville domain in M bounded by Y and $M^+ = M\backslash(M^-\backslash\partial M^-)$. Let SM^- and SM^+ be the positive and negative symplectic completion of M^- and M^+, respectively. Given $(J^\tau)_{\tau\in[0,\infty)}$, there is a unique almost complex structure J^-, J^Y and J^+ on SM^-, SY and SM^+, respectively, such that $(M^-, J^\tau|_{M^-})$, $(N(Y), J^\tau|_{N(Y)})$ and $(M^+, J^\tau|_{M^+})$ converges to (SM^-, J^-), (SY, J^Y) and (SM^+, J^+), respectively, as τ goes to infinity. More details about this splitting procedure can be found in [32, Section 3].

Remark 3.2 There is a variant for being adjusted to $N(Y)$. For a fixed number $R \geqslant 0$, we call a smooth family of compatible almost complex structure $(J^\tau)_{\tau\in[3R,\infty)}$ on M is R*-adjusted to* $N(Y)$ if (3.2) is satisfied but the property of $(J^\tau)'$ is replaced by the following conditions.

$$\begin{cases} (J^\tau)'|_{[-(\tau+\epsilon-2R),\tau+\epsilon-2R]\times Y} \text{ is cylindrical for all } \tau \\ (J^\tau)'|_{(-\epsilon,\epsilon)\times Y} = (\Phi_{N(Y)})_*(J^0|_{N(Y)}) \text{ for all } \tau \\ (J^{\tau_1})'|_{(-(\tau_1+\epsilon),-(\tau_1+\epsilon-2R)]\times Y} = (\phi^-_{\tau_1,\tau_2})_*(J^{\tau_2})'|_{(-(\tau_2+\epsilon),-(\tau_2+\epsilon-2R)]\times Y} \text{ for all } \tau_1, \tau_2 \\ (J^{\tau_1})'|_{[\tau_1+\epsilon-2R,\tau_1+\epsilon)\times Y} = (\phi^+_{\tau_1,\tau_2})_*(J^{\tau_2})'|_{[\tau_2+\epsilon-2R,\tau_2+\epsilon)\times Y} \text{ for all } \tau_1, \tau_2 \end{cases}$$

where $\phi^-_{\tau_1,\tau_2} : (-(\tau_2+\epsilon), -(\tau_2+\epsilon-2R)] \times Y \to (-(\tau_1+\epsilon), -(\tau_1+\epsilon-2R)] \times Y$ and $\phi^+_{\tau_1,\tau_2} : [\tau_2+\epsilon-2R, \tau_2+\epsilon) \times Y \to [\tau_1+\epsilon-2R, \tau_1+\epsilon) \times Y$ are the r-translation.

When $R = 0$, being R-adjusted to $N(Y)$ is the same as being adjusted to $N(Y)$. For $R > 0$, we can also define $J^\pm$, J^Y accordingly.

In this case, J^+ (resp. J^-) are cylindrical over the end $(-\infty, -2R] \times \partial M^+ \subset SM^+$ (resp. $[2R, \infty) \times \partial M^- \subset SM^-$).

Let L be a Lagrangian submanifold in M such that $L \cap N(Y) = (-\epsilon, \epsilon) \times \Lambda$ for some (possibly empty) Legendrian submanifold Λ. Let $L^\pm := L \cap M^\pm$. We define $SL^- = L^- \cup (\mathbb{R}_{\geqslant 0} \times \Lambda) \subset SM^-$ and $SL^+ = L^+ \cup (\mathbb{R}_{\leqslant 0} \times \Lambda) \subset SM^+$ which are the cylindrical extensions of L^- and L^+ with respect to the symplectic completion. We denote $\mathbb{R} \times \Lambda \subset SY$ by $S\Lambda$.

The main ingredient we needed from [32] is the following compactness result in symplectic field theory.

Theorem 3.3 ([32] Theorem 10.3 and Section 11.3; see also [33]) *Let L_j, $j = 0, \dots, d$ be a collection of embedded exact Lagrangian submanifolds in M such that $L_i \pitchfork L_j$ for all $i \neq j$. Let $(Y, \alpha) \subset M$ be a contact type hypersurface and $(N(Y), \omega|_{N(Y)}) \cong ((-\epsilon, \epsilon) \times Y, d(e^r\alpha))$ be a neighborhood of Y such that $L_i \cap N(Y) = (-\epsilon, \epsilon) \times \Lambda_i$ for some (possibly empty) Legendrian submanifold Λ_i of Y.*

Let J^τ be a smooth family of almost complex structures R-adjusted to $N(Y)$. Let $x_0 \in CF(L_0, L_d)$ and $x_j \in CF(L_{j-1}, L_j)$ for $j = 1, \dots, d$. If there exists a sequence $\{\tau_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} \tau_k = \infty$, and a sequence $u_k \in \mathcal{M}^{J^{\tau_k}}(x_0; x_d, \dots, x_1)$, then u_k converges to a holomorphic building $u_\infty = \{u_v\}_{v\in V(\mathcal{T})}$ in the sense of [32].

We remark that each J^τ above is a domain independent almost complex structure (see Remark 2.1) and we do not need to assume u_k to be transversally cut out to apply Theorem 3.3.

The rest of this subsection is devoted to the description/definition of $u_\infty = \{u_v\}_{v\in V(\mathcal{T})}$ in Theorem 3.3. The definition is quite well-known so we only give a quick review and introduce necessary notations along the way.

First, $\mathcal{T}$ is a tree with $d+1$ semi-infinite edges and one of them is distinguished which is called the root. The other semi-infinite edges are ordered from 1 to d and called the leaves. Let $V(\mathcal{T})$ be the set of vertices of T. For each $v \in V(\mathcal{T})$, we have a punctured Riemannian surface Σ_v. If $\partial\Sigma_v \neq \varnothing$, there is a distinguished boundary puncture which is denoted by ξ_0^v. After filling the punctures of Σ_v, it is a topological disk so we can label the other boundary punctures of Σ_v by $\xi_1^v, \dots \xi_{d_v}^v$ counterclockwise along the boundary, where $d_v + 1$ is the number of boundary punctures of Σ_v. Let $\partial_j\Sigma_v$ be the component of $\partial\Sigma_v$ that goes from ξ_j^v to ξ_{j+1}^v for $j = 0, \dots, d_v - 1$, and $\partial_{d_v}\Sigma_v$ be the component of $\partial\Sigma_v$ that goes from $\xi_{d_v}^v$ to ξ_0^v. If $\partial\Sigma_v = \varnothing$, then Σ_v is a sphere after filling the punctures.

There is a bijection f_v from the punctures of Σ_v to the edges in $\mathcal{T}$ adjacent to v. Moreover, $f_v(\xi_0^v)$ is the edge closest to the root of $\mathcal{T}$ among edges adjacent to v. If v, v' are two distinct vertices adjacent to e, then $f_v^{-1}(e)$ and $f_{v'}^{-1}(e)$ are either both boundary punctures or both interior punctures. We call e a *boundary edge* (resp. an *interior edge*) if $f_v^{-1}(e)$ is a boundary (resp. an interior) puncture. We can glue $\{\Sigma_v\}_{v\in V(\mathcal{T})}$ along the punctures according to the edges and $\{f_v\}_{v\in V(\mathcal{T})}$ (i.e. Σ_v is glued with $\Sigma_{v'}$ by identifying $f_v^{-1}(e)$ with $f_{v'}^{-1}(e)$ if v, v' are two distinct vertices adjacent to e). After gluing, we will get back S, the domain of u_k, topologically. Therefore, there is a unique way to assign Lagrangian labels to $\partial\Sigma_v$ such that it is compatible with gluing and coincides with that on ∂S after gluing all Σ_v together. We denote the resulting Lagrangian label on $\partial_j\Sigma_v$ by $L_{v,j}$.

There is a level function $l_\mathcal{T} : V(\mathcal{T}) \to \{0, \dots, n_\mathcal{T}\}$ for some positive integer $n_\mathcal{T}$. If $l_\mathcal{T}(v) = 0$, then $u_v : \Sigma_v \to SM^-$ is a J^--holomorphic curve such that $u_v(\partial_j\Sigma_v) \subset SL_{v,j}^-$. If $l_\mathcal{T}(v) = 1, \dots, n_\mathcal{T} - 1$, then $u_v : \Sigma_v \to SY$ is a J^Y-holomorphic curve such that $u_v(\partial_j\Sigma_v) \subset S\Lambda_{v,j}$. If $l_\mathcal{T}(v) = n_\mathcal{T}$, then $u_v : \Sigma_v \to SM^+$ is a J^+-holomorphic curve such that $u_v(\partial_j\Sigma_v) \subset SL_{v,j}^+$.

If $v \neq v'$ are adjacent to the same edge e in $\mathcal{T}$, then $|l_\mathcal{T}(v) - l_\mathcal{T}(v')| \leqslant 1$. If $l_\mathcal{T}(v) + 1 = l_\mathcal{T}(v')$ and e is a boundary (resp. interior) edge, then there is a Reeb chord (resp. orbit) which is the positive asymptote of u_v at $f_v^{-1}(e)$, and the negative asymptote of $u_{v'}$ at $f_{v'}^{-1}(e)$ (see Convention 3.6). If $l_{\mathcal{T}}(v) = l_{\mathcal{T}}(v')$, then e is necessarily a boundary edge, $l_{\mathcal{T}}(v) = l_{\mathcal{T}}(v') \in \{0, n_{\mathcal{T}}\}$ and $u_v, u_{v'}$ converges to the same Lagrangian intersection point at $f_v^{-1}(e)$, $f_{v'}^{-1}(e)$, respectively. If e is the j^{th} semi-infinite edge adjacent to v, then u_v is asymptotic to x_j at $f_v^{-1}(e)$.

Finally, for each $j = 1, \dots, n_{\mathcal{T}} - 1$, there is at least one $v \in V(\mathcal{T})$ such that $l_{\mathcal{T}}(v) = j$ and u_v is not a *trivial cylinder* (i.e. u_v is not a map $\mathbb{R} \times [0, 1] \to SY$ or $\mathbb{R} \times S^1 \to SY$) such that

$$u_v(s, t) = (f_r(s), f_Y(t)) \in \mathbb{R} \times Y \tag{3.3}$$

for some f_r, f_Y). We use $\mathcal{M}^{J^\infty}(x_0; x_d, \dots, x_1)$ to denote the set of such holomorphic buildings.

Remark 3.4 From this point on, Theorem 3.3 will play a major role in analyzing holomorphic curves.

It is important to note that, the domain of a holomorphic building under our consideration can always be glued up into a smooth disk with boundary, which is the domain for J^τ when $\tau < \infty$.

For our application, we assume **every** holomorphic disks $u : \Sigma \to M$ which undergoes an SFT-stretching process must have pairwisely distinct Lagrangian boundary conditions on different components of $\partial\Sigma$ when $\tau < \infty$ throughout the rest of the paper. The reason we impose this condition is because we use a perturbation scheme in defining the Fukaya category, therefore, Lagrangian boundary conditions on two different connected components of $\partial\Sigma$ are never the same Lagrangian. This will play a key role in our configuration analysis of the buildings.

Let

- V^{core} be the set of vertices $v \in V(\mathcal{T})$ such that more than one Lagrangian appears in the Lagrangian labels of $\partial\Sigma_v$.
- V^{∂} be the set of vertices $v \in V(\mathcal{T})$ such that there is only one Lagrangian appears in the Lagrangian labels of $\partial\Sigma_v$.
- V^{int} be the set of vertices $v \in V(\mathcal{T})$ such that $\partial\Sigma_v = \emptyset$.

In particular, we have $V(\mathcal{T}) = V^{core} \sqcup V^{\partial} \sqcup V^{int}$. Let $\mathcal{T}^{core}$, $\mathcal{T}^{\partial}$ and $\mathcal{T}^{int}$ be the subgraphs of $\mathcal{T}$, which consists of vertices V^{core}, V^{∂} and V^{int}, and edges adjacent to their respective vertices (see Fig. 1 for an example). Note that these three subtrees *could* have overlaps.

Lemma 3.5 *The graphs $\mathcal{T}^{(1)} := \mathcal{T}^{core} \backslash \mathcal{T}^{int}$ and $\mathcal{T}^{(2)} := (\mathcal{T}^{core} \cup \mathcal{T}^{\partial}) \backslash \mathcal{T}^{int}$ are planar trees. In particular, they are connected.*

Proof Let G be a minimal subtree of $\mathcal{T}$ containing $\mathcal{T}^{(1)}$. If there is a vertex v in G such that $v \in V^{int}$, then it would imply that S, the domain of u_k, is not a disk. If there is a vertice v in G such that $v \in V^{\partial}$, then it would imply that there is a Lagrangian that appears more than once in the Lagrangian label of ∂S. Both of these situations are not possible.

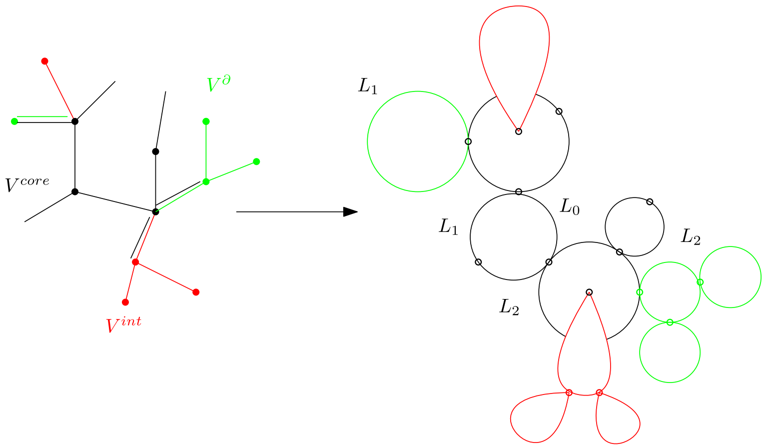

Fig. 1 A tree $\mathcal{T}$ with 2 leaves. Black dots: elements in V^{core}; Green dots: elements in V^{∂}; Red dots: elements in V^{int}; Black tree: $\mathcal{T}^{core}\backslash(\mathcal{T}^{\partial} \cup \mathcal{T}^{int})$; Green sugbraph: $\mathcal{T}^{\partial}\backslash\mathcal{T}^{int}$; Red subgraph: $\mathcal{T}^{int}$ (Color figure online)

Similarly, let G' be the smallest subtree of $\mathcal{T}$ containing $\mathcal{T}^{(2)}$. If there is a vertice v in G' such that $v \in V^{int}$, then it would imply that S is not a disk and we get a contradiction.

As a result, $G = \mathcal{T}^{(1)}$ and $G' = \mathcal{T}^{(2)}$ so both $\mathcal{T}^{(1)}$ and $\mathcal{T}^{(2)}$ are trees.

The fact that $\mathcal{T}^{(1)}$ and $\mathcal{T}^{(2)}$ are planar follows from the fact that we can order the boundary punctures of Σ_v, for $v \in V^{core} \cup V^{\partial}$, in a way that is compatible with the boundary orientation. $\square$

Convention 3.6 *We need to explain the convention of strip-like ends and cylindrical ends we use for punctures of Σ_v. Let e be an edge in $\mathcal{T}$ and $v \neq v'$ are the vertices adjacent to e.*

First assume that $l_{\mathcal{T}}(v) + 1 = l_{\mathcal{T}}(v')$. If e is a boundary (resp. interior) edge, we use an outgoing/positive strip-like end (2.5) (resp. cylindrical end) for $f_v^{-1}(e)$, where an ***outgoing/positive cylindrical end*** *for $f_v^{-1}(e)$ is a holomorphic embedding of $\epsilon_{v,e} : \{z = s\exp(\sqrt{-1}t) \in \mathbb{C} | s \geqslant 1\} \to \Sigma_v$ such that $\lim_{|z|\to\infty} \epsilon_{v,e}(z) = f_v^{-1}(e)$. With respect to coordinates given by the strip-like (resp. cylindrical) end $\epsilon_{v,e}$, we have*

$$\begin{cases} \lim_{s\to\infty} \pi_Y(u_v(\epsilon_{v,e}(s,t))) = x(Tt) \text{ (resp. } \gamma(Tt)) \\ \lim_{s\to\infty} \pi_{\mathbb{R}}(u_v(\epsilon_{v,e}(s,t))) = \infty \end{cases} \tag{3.4}$$

for some Reeb chord x (resp. orbit γ) and some $T > 0$, where $\pi_Y, \pi_{\mathbb{R}}$ are the projection from SY to the two factors. In this case, we call x (resp. γ) the positive asymptote of u_v at $f_v^{-1}(e)$.

On the other hand, we use an incoming/negative strip-like end (2.6) (resp. cylindrical end) for $f_{v'}^{-1}(e)$, where an ***incoming/negative cylindrical end*** *for $f_{v'}^{-1}(e)$ is a holomorphic embedding of $\epsilon_{v',e} : \{z = s\exp(\sqrt{-1}t) \in \mathbb{C} | 0 < s \leqslant 1\} \to \Sigma_{v'}$ such* *that* $\lim_{|z|\to 0} \epsilon_{v',e}(z) = f_{v'}^{-1}(e)$. *With respect to coordinates given by the strip-like (resp. cylindrical) end* $\epsilon_{v',e}$, *we have*

$$\begin{cases} \lim_{s\to 0} \pi_Y(u_v(\epsilon_{v',e}(s,t))) = x(Tt) \; (\textit{resp. } \gamma(Tt)) \\ \lim_{s\to 0} \pi_{\mathbb{R}}(u_v(\epsilon_{v',e}(s,t))) = -\infty \end{cases} \tag{3.5}$$

or some Reeb chord x (*resp. orbit* γ) *and some* $T > 0$. *In this case, we call* x (*resp.* γ) *the negative asymptote of* $u_{v'}$ *at* $f_{v'}^{-1}(e)$.

If $l_{\mathcal{T}}(v) = l_{\mathcal{T}}(v')$ *and, say* v *is closer to the root of* $\mathcal{T}$ *than* v', *then we use an outgoing/positive strip-like end for* $f_v^{-1}(e)$ *and an incoming/negative strip-like end for* $f_{v'}^{-1}(e)$. *Similarly, the intersection point that they are asymptotic to is the positive asymptote of* u_v *at* $f_v^{-1}(e)$ *and the negative asymptote of* $u_{v'}$ *at* $f_{v'}^{-1}(e)$.

3.2 Gradings

Let $P \subset (M, \omega, \theta)$ be a Lagrangian submanifold which satisfies (2.58). In particular, $H^1(P, \mathbb{R}) = 0$ and P is an exact Lagrangian. The round metric on S^n descends to a Riemannian metric on P. Let U be a Weinstein neighborhood of P and we identify ∂U with the set of covectors of P having a common small fixed norm. Without loss of generality, we can assume that $\theta|_U = \theta_{T^*P}$, where θ_{T^*P} is the standard Liouville one-form on T^*P. Let $\alpha_0 := \theta|_{\partial U}$ be the standard contact form on ∂U. Eventually, we will apply Theorem 3.3 along a perturbation $(\partial U)'$ of ∂U. Since $((\partial U)', \theta|_{(\partial U)'}) \simeq (\partial U, \alpha')$ for a perturbation α' of α_0, we will need to understand the Reeb dynamics of α'. Therefore, it is helpful to explain the Reeb dynamics of $(\partial U, \alpha_0)$ first. We assume $\Lambda_i := L_i \cap \partial U$ are (possibly empty) unions of cospheres at points of P. There are four types of asymptotes that can appear for u_v near the punctures.

1. Lagrangian intersection points between $SL_i^{\pm}$ and $SL_j^{\pm}$ in $SM^{\pm}$,
2. Reeb chords from Λ_i to Λ_j in Y for $i \neq j$,
3. Reeb chords from Λ_i to itself in Y, and
4. Reeb orbits in Y

We want to discuss the grading for each of these types.

3.2.1 Type one

Let Ω be the nowhere-vanishing section of $(\Lambda_{\mathbb{C}}^{top} T^*M)^{\otimes 2}$ which equals to 1 with respect to the chosen trivialization (see Standing Assumption). For a Lagrangian subspace $V \subset T_pM$ and a choice of basis $\{X_1, \ldots, X_n\}$ of V, we define

$$Det_{\Omega}(V) := \frac{\Omega(X_1, \ldots, X_n)}{\|\Omega(X_1, \ldots, X_n)\|} \in S^1 \tag{3.6}$$

which is independent of the choice of basis. A $\mathbb{Z}$**-grading of** L_i is a continuous function $\theta_{L_i} : L_i \to \mathbb{R}$ such that $e^{2\pi\sqrt{-1}\theta_{L_i}(p)} = Det_{\Omega}(T_pL_i)$ for all $p \in L_i$.

At each transversal intersection point $x \in L_i \cap L_j$, we have two graded Lagrangian planes $T_x L_i$, $T_x L_j$ inside $T_x M$. The grading of x as a generator of $CF(L_i, L_j)$ is given by the Maslov grading from $T_x L_i$ to $T_x L_j$ which is

$$|x| = \iota(T_x L_i, T_x L_j) := n + \theta_{L_j}(x) - \theta_{L_i}(x) - 2Angle(T_x L_i, T_x L_j) \tag{3.7}$$

where $Angle(T_x L_i, T_x L_j) = \sum_{j=1}^{n} \beta_j$ and $\beta_j \in (0, \frac{1}{2})$ are such that there is a unitary basis $u_1, \ldots, u_n$ of $T_x L_i$ satisfying $T_x L_j = Span_{\mathbb{R}}\{e^{2\pi\sqrt{-1}\beta_j} u_j\}_{j=1}^{n}$. If we regard x as an element in $CF(L_j, L_i)$, then we have $\iota(T_x L_j, T_x L_i) = n - \iota(T_x L_i, T_x L_j)$.

Convention 3.7 *For a generator $x \in CF(L_i, L_j)$, we use $x^\vee$ to denote the generator of $CF(L_j, L_i)$ which represents the same intersection point as x. Therefore, we have $|x| = n - |x^\vee|$.*

Since $SM^- = T^*P$ and $w_2(P) = 0$ and $c_1(T^*P) = 0$, there is a preferred choice of trivialization of $(\Lambda_{\mathbb{C}}^{top} T^*SM^-)^{\otimes 2}$ such that the grading functions on cotangent fibers and the zero section are constant functions (see [29]). Without loss of generality, we can assume that the restriction to M^- of the choice of trivialization of $(\Lambda_{\mathbb{C}}^{top} T^*M)^{\otimes 2}$ we picked coincides with that of $(\Lambda_{\mathbb{C}}^{top} T^*SM^-)^{\otimes 2}$. We call that the cotangent fibers and the zero section are in *canonical relative grading* if the following holds:

$$CF(P, T_q^* P) \text{ is concentrated at degree } 0 \tag{3.8}$$

for all $q \in P$.

We refer readers to [29], [2, Section 11, 12] for more about Maslov gradings.

3.2.2 Type two

In general, if we have a Reeb chord $x = (x(t))_{t\in[0,1]}$ from Λ_0 to Λ_1 in a contact manifold (Y, α) such that $S\Lambda_i$ are graded Lagrangians in SY for both i, we will assign a grading to x by regarding x as a Hamiltonian chord between graded Lagrangians $S\Lambda_0$ and $S\Lambda_1$ in the symplectic manifold SY as follows: There is an appropriate Hamiltonian H in SY that depends only on the radial coordinate r such that the Reeb vector field R_α in Y coincides with the restriction of the Hamiltonian vector field X_H to $\{0\} \times Y$. Let ϕ^H be the time-one flow of H. We identify $x(t) \in Y$ with $(0, x(t)) \in SY$ so x is a H-Hamiltonian chord. We have graded Lagrangian subspaces $(\phi^H)_* T_{x(0)} S\Lambda_0$ and $T_{x(1)} S\Lambda_1$ in $T_{x(1)} SY$. Let

$$K_x := (\phi^H)_*(T_{x(0)} S\Lambda_0) \cap T_{x(1)} S\Lambda_1 \tag{3.9}$$

The grading $|x|$ of x is defined to be

$$|x| = \iota((\phi^H)_*(T_{x(0)} S\Lambda_0)/K_x, T_{x(1)} S\Lambda_1/K_x) \tag{3.10}$$

where the Maslov grading [see (3.7)] is computed in the symplectic vector space $T_{x(1)}M/(K_x + J(K_x))$. More details about Maslov gradings assigned to non-transversally intersecting graded Lagrangian subspaces can be found in [21, Section 4.1], for example.

Now, we go back to our situation and assume x is a Reeb chord from Λ_i to Λ_j in $(\partial U, \alpha_0)$. Since L_i is graded, $S\Lambda_i$ has a grading function in $S(\partial U)$ inherited from L_i. The computation of $|x|$ is done in the literature (e.g. [34,35], where they indeed proved $HW(T_{\mathbf{q}_i}) \cong k[u]$ for $|u| = -(n-1)$) and we recall it here.

Without loss of generality, we assume Λ_i and Λ_j are connected. Let $q_i, q_j \in P$ be such that $T^*_{q_i}P \cap \partial U = \Lambda_i$ and $T^*_{q_j}P \cap \partial U = \Lambda_j$. We equip the cotangent fibers and P with the canonical relative grading [see (3.8)]. The grading functions of L_i and L_j differs from the grading functions of $T^*_{q_i}P$ and $T^*_{q_j}P$ near Λ_i and Λ_j, respectively, by an integer. In the following, we will assume the grading functions coincide and the actual $|x|$ can be recovered by adding back the integral differences of the grading functions.

Let $\mathbf{q}_i \in \mathbf{P}$ be a lift of q_i. Each Reeb chord x from Λ_i to Λ_j corresponds to a geodesic from q_i to q_j, which can be lifted to a geodesic $\mathbf{x}$ from $\mathbf{q}_i$ to a point $\mathbf{q}_j \in \mathbf{P}$ such that $\pi(\mathbf{q}_j) = q_j$. If

$$\mathbf{q}_j \text{ is not the antipodal point of } \mathbf{q}_i \text{ and } \mathbf{q}_j \neq \mathbf{q}_i \tag{3.11}$$

then there is a unique closed geodesic (assumed to have length 2π) passing through $\mathbf{q}_i$ and $\mathbf{q}_j$. Therefore, for each interval $I_k = (k\pi, (k+1\pi))$, $k \in \mathbb{N}$, there is a unique geodesic from $\mathbf{q}_i$ to $\mathbf{q}_j$ with length lying inside I_k. If the length of $\mathbf{x}$ lies in I_k, then

$$|x| = -k(n-1) \tag{3.12}$$

For generic q_i, q_j, every lifts $\mathbf{q}_i, \mathbf{q}_j$ of q_i, q_j satisfies (3.11).

3.2.3 Type three

There are four kinds of Reeb chords from Λ_i to itself. First, if x is a Reeb chord from one connected component of Λ_i to a different one, then the computation of $|x|$ reduces to the Type two (Sect. 3.2.2). For the remaining three kinds, we assume $\Lambda_i = T^*_{q_i}P \cap \partial U$, i.e. it has exactly one connected component. Let $\mathbf{q}_i$ be a lift of q_i and $\mathbf{x} : [0,1] \to \mathbf{P}$ be the lift of the geodesic such that $\mathbf{x}(0) = \mathbf{q}_i$, $\mathbf{q}'_i := \mathbf{x}(1)$ and $\pi(\mathbf{q}'_i) = q_i$. The three possibilities are

1. $\mathbf{q}_i, \mathbf{q}'_i$ satisfy (3.11) (with $\mathbf{q}'_i$ replacing $\mathbf{q}_j$), or
2. $\mathbf{q}'_i$ is the antipodal point of $\mathbf{q}_i$, or
3. $\mathbf{q}_i = \mathbf{q}'_i$

For the first case, the computation of $|x|$ reduces to the previous one again (Sect. 3.2.2). For the second and the third cases, we have [see (3.9) for the meaning of K_x]

$$K_x = T_{x(1)}L_i \tag{3.13}$$

$$|x| = \iota((\phi^H)_* T_{x(0)} L_i / K_x, T_{x(1)} L_j / K_x) = -k(n-1) \tag{3.14}$$

where $k\pi$ is the length of $\mathbf{x}$, and the term $-k(n-1)$ is exactly the (integral) difference of the values of the grading functions at $(\phi^H)_* T_{x(0)} L_i$ and $T_{x(1)} L_i$ as graded Lagrangian planes.

We want to point out that in the second and third cases x lies in a Morse–Bott family S_x of Reeb chords from Λ_i to itself and $\dim(S_x) = n-1$.

3.2.4 Type four

Reeb orbits of ∂U are graded by the Robbin–Salamon index [36] (see also [37, Section 5]), which is a generalization of the Conley-Zehnder index to the degenerated case. To define the Robbin–Salamon index of a Reeb orbit γ, we need to pick a symplectic trivialization Φ_γ of ξ along γ subject to the following compatibility condition: together with the obvious trivialization of $\mathbb{R}\langle \partial_r, R_{\alpha_0}\rangle$, Φ_γ gives a symplectic trivialization of TM along γ, and hence a trivialization of $(\Lambda_{\mathbb{C}}^{top} T^*M)^{\otimes 2}$ along γ. The compatibility condition is that the induced trivialization of $(\Lambda_{\mathbb{C}}^{top} T^*M)^{\otimes 2}$ along γ coincides with the trivialization of $(\Lambda_{\mathbb{C}}^{top} T^*M)^{\otimes 2}$ we picked in the beginning of Sect. 2. One may show that there is Φ_γ satisfying the compatibility condition.

We can now define a path of symplectic matrices $(\Phi_t)_{t\in[0,1]}$ given by $\Phi_t := (\phi_t^R)_* : \xi_{\gamma(0)} \to \xi_{\gamma(t)} \simeq \xi_{\gamma(0)}$, where ϕ_t^R is the time-t flow generated by R_{α_0} and the last isomorphism is given by Φ_γ. We can assign the Robbin–Salamon index for Φ_t as follows: first, we isotope (relative to end points) Φ_t to a path of symplectic matrices Φ'_t such that $\ker(\Phi'_t - Id) \neq 0$ happens at finitely many times $t = t_1, \ldots, t_k$ and for each t_j, the crossing form $J\frac{d}{dt}|_{t=t_j}(\Phi'_t)$ is non-degnerate on $\ker(\Phi'_{t_j} - Id)$. The signature of $J\frac{d}{dt}|_{t=t_j}(\Phi'_t)$ is denoted by $\sigma(t_j)$ and the Robbin–Salamon index is defined by

$$\mu_{RS}(\Phi_t) := \frac{1}{2}\sigma(0) + \sum_{j=1}^{k} \sigma(t_j) + \frac{1}{2}\sigma(1) \tag{3.15}$$

where $\sigma(1)$ is defined to be zero if Φ_1 is invertible. The index is independent of the choice of Φ'_{t_j}. The Robbin–Salamon index of γ with respect to the trivialization Φ_γ is

$$\mu_{RS}(\gamma) := \mu_{RS}(\Phi_t) \tag{3.16}$$

Any two choices of Φ_γ satisfying the compatibility condition would give the same index.

There are two kinds of Reeb orbits γ in ∂U, namely, contractible in U or not. We are only interested in the case that γ is contractible in U, which means that it can be lifted to a Reeb orbit in $\partial\mathbf{U}$ that is contractible in $\mathbf{U}$. The lifted Reeb orbit corresponds to a geodesic loop l_γ in $\mathbf{P}$. The Robbin–Salamon index $\mu_{RS}(\gamma)$ is computed in [38, Lemma 7] (the proof there can be directly generalized to all n)

$$\mu_{RS}(\gamma) = 2k(n-1) \tag{3.17}$$

where k is the covering multiplicity of l_γ with respect to the simple geodesic loop, or equivalently, $2k\pi$ is the length of l_γ.

We want to point out that γ lies in a Morse–Bott family S_γ of (unparametrized) Reeb orbits and $\dim(S_\gamma) = 2n-2$.

3.3 Dimension formulae

In this section, we first review the virtual dimension formula from [37], where the domain of the pseudo-holomorphic map only has interior punctures. Then we consider the case that the domain only has boundary punctures, and finally obtain the general formula by gluing.

Let $(Y^\pm, \alpha^\pm)$ be contact manifolds with contact forms $\alpha^\pm$. We assume that every Reeb orbit γ of $Y^\pm$ lies in a Morse–Bott family S_γ of (unparametrized) Reeb orbits. Let (X, ω_X) be a symplectic manifold such that there exists a compact set $K_X \subset X$ and $T_X \in \mathbb{R}_{>0}$ so that $(X \backslash K_X, \omega_X|_{X\backslash K_X})$ is the disjoint union of the ends $([T_X, \infty) \times Y^+, d(e^r\alpha^+))$ and $((-\infty, -T_X] \times Y^-, d(e^r\alpha^-))$. In this case, we have

Lemma 3.8 ([37], Corollary 5.4) *Let Σ be a punctured Riemannian surface of genus g and $\partial\Sigma = \varnothing$. Let J be a compatible almost complex structure on X that is cylindrical over the ends. Let $u : \Sigma \to X$ be a J-holomorphic map with positive asymptotes $\{\gamma_j^+\}_{j=1}^{s_+}$ and negative asymptotes $\{\gamma_j^-\}_{j=1}^{s_-}$ (see Convention 3.6).*

Then the virtual dimension of u is given by

$$\begin{aligned} \mathrm{virdim}(u) = (n-3)(2-2g-s^+-s^-) + \sum_{j=1}^{s^+} \mu_{RS}(\gamma_j^+) \\ - \sum_{j=1}^{s^-} \mu_{RS}(\gamma_j^-) + \frac{1}{2}\sum_{j=1}^{s^+} \dim(S_{\gamma_j^+}) \\ + \frac{1}{2}\sum_{j=1}^{s^-} \dim(S_{\gamma_j^-}) + 2c_1^{rel}(TX)([u]) \end{aligned} \tag{3.18}$$

where $2c_1^{rel}(TX)([u])$ is the relative first Chern class computed with respect to the fixed symplectic trivializations along the Reeb orbits that we chose to compute μ_{RS} (see Sect. 3.2.4).

Sketch of proof As explained in Sect. 3.2.4, the trivialization $\Phi_{\gamma_j^\pm}$ of ξ along $\gamma_j^\pm$ determines a path of symplectic matrices $\Phi_t^{\pm,j}$. We can trivialize TX along $\gamma_j^\pm$ using $\Phi_{\gamma_j^\pm}$ by adding the invariant directions ∂_r, $R_\alpha^\pm$. The corresponding path of symplectic matrices become $\overline{\Phi}_t^{\pm,j} = \Phi_t^{\pm,j} \oplus I_{2\times 2}$, where $I_{2\times 2}$ is the 2 by 2 identity matrix. By additivity property of μ_{RS}, we have $\mu_{RS}(\overline{\Phi}_t^{\pm,j}) = \mu_{RS}(\Phi_t^{\pm,j}) + \mu_{RS}(I_{2\times 2}) = \mu_{RS}(\Phi_t^{\pm,j})$.

If $\ker(\overline{\Phi}_1^{\pm,j} - Id) = 0$ (which is never the case) for all $\gamma_j^\pm$, then the index of u is given by

$$\mathrm{ind}(u) = n(2-2g-s^+-s^-) + \sum_{j=1}^{s^+}\mu_{RS}(\gamma_j^+) - \sum_{j=1}^{s^-}\mu_{RS}(\gamma_j^-) + 2c_1^{rel}(TX)([u]) \tag{3.19}$$

If $\ker(\overline{\Phi}_1^{\pm,j} - Id) \neq 0$, then it contributes $\dim(\ker(\overline{\Phi}_1^{\pm,j} - Id)) = \dim(S_{\gamma_j^\pm}) + 2$ (resp. 0) to $\mathrm{ind}(u)$ when $\gamma_j^\pm$ is a positive (resp. negative) asymptote. However, the definition of μ_{RS} already takes into account $\frac{1}{2}\dim(\ker(\overline{\Phi}_1^{\pm,j} - Id))$ so we have

$$\mathrm{ind}(u) = n(2-2g-s^+-s^-) + \sum_{j=1}^{s^+}(\mu_{RS}(\gamma_j^+) + \frac{1}{2}(\dim(S_{\gamma_j^+})+2)) \tag{3.20}$$

$$-\sum_{j=1}^{s^-}(\mu_{RS}(\gamma_j^-) - \frac{1}{2}(\dim(S_{\gamma_j^-})+2)) + 2c_1^{rel}(TX)([u]) \tag{3.21}$$

$$= n(2-2g-s^+-s^-) + \sum_{j=1}^{s^+}\mu_{RS}(\gamma_j^+) - \sum_{j=1}^{s^-}\mu_{RS}(\gamma_j^-) + \frac{1}{2}\sum_{j=1}^{s^+}\dim(S_{\gamma_j^+}) + \frac{1}{2}\sum_{j=1}^{s^-}\dim(S_{\gamma_j^-}) + (s^+ + s^-) + 2c_1^{rel}(TX)([u]) \tag{3.22}$$

Finally, to obtain the virtual dimension, we need to add the dimension of the Teichmüller space that Σ lies, which is $6g-6+2(s^+ + s^-)$. It gives the formula (3.18). □

We note that Lemma 3.8 still holds when $Y^\pm = \varnothing$, where the corresponding $s^\pm = 0$.

Example 3.9 The virtual dimension of $u : \mathbb{C} \to T^*S^n$ with the puncture asymptotic to a simple Reeb orbit is given by

$$\mathrm{virdim}(u) = (n-3)(2-2(0)-1-0) + 2(n-1) + \frac{1}{2}(2n-2) = 4n-6 \tag{3.23}$$

because $c_1^{rel}(TT^*S^n) = 0$. When $n=2$, we have $\mathrm{virdim}(u) = 2$ which is obtained in [38, Lemma 7].

Now, we consider the relative setting. A Lagrangian cobordism L in X is a Lagrangian such that there exists $T > T_X$ so that $L \cap (-\infty, -T] \times Y^- = (-\infty, -T] \times \Lambda^-$ and $L \cap [T, \infty) \times Y^+ = [T, \infty) \times \Lambda^+$ for some Legendrian submanifolds $\Lambda^\pm$ in $Y^\pm$. Let L_0, L_1 be exact Lagrangian cobordisms such that $L_0 \pitchfork L_1$.

We assume that every Reeb chord x from $\Lambda_0^{\pm}$ to $\Lambda_1^{\pm}$ lies in a Morse–Bott family S_x of Reeb chords. In this case, we define [see (3.9)]

$$\mathrm{mb}(x) = \dim(S_x) + 1 = \dim(K_x) \tag{3.24}$$

If $x \in L_0 \cap L_1$, we define $\mathrm{mb}(x) = 0$. The reader should note that the discrepancy between $mb(x)$ and $dim(S_x)$ comes from the $\mathbb{R}$-direction of symplectizations. As always, we assume that L_0, L_1 are $\mathbb{Z}$-graded so all elements in $L_0 \cap L_1$, and all Reeb chords from $\Lambda_0^{\pm}$ to $\Lambda_1^{\pm}$ are graded (see Sect. 3.2).

Lemma 3.10 *Let $S \in \mathcal{R}^{d+1}$ be equipped with Lagrangian labels L_j on $\partial_j S$, where each L_j is a Lagrangian cobordism. Let J be a compatible almost complex structure on X that is cylindrical on the ends. Let $u : S \to X$ be a J-holomorphic map with positive asymptotes $\{x_j^+\}_{j=1}^{r^+}$ and negative asymptotes $\{x_j^-\}_{j=1}^{r^-}$ such that $u(\partial_j S) \subset L_j$. Assume all asymptotes are Morse–Bott, then the virtual dimension of u is given by*

$$\mathrm{virdim}(u) = n(1-r^-) + \sum_{j=1}^{r^-}(\iota(x_j^-) + \mathrm{mb}(x_j^-)) - \sum_{j=1}^{r^+}\iota(x_j^+) + (r^- + r^+ - 3) \tag{3.25}$$

Sketch of proof When all $x_j^{\pm}$ are Lagrangian intersection points, then the index of u is given by (see [2, Proposition 11.13])

$$\mathrm{ind}(u) = n(1-r^-) + \sum_{j=1}^{r^-}\iota(x_j^-) - \sum_{j=1}^{r^+}\iota(x_j^+) \tag{3.26}$$

If $x_j^{\pm}$ is a Reeb chord, then the intersection of the graded Lagrangian subspaces $K_{x_j^{\pm}}$ is non-zero. Similar to the proof of Lemma 3.8, there are extra contributions to $\mathrm{virdim}(u)$ from the asymptotes. This time, $x_j^{\pm}$ contributes $\dim(K_{x_j^{\pm}}) = \mathrm{mb}(x_j^{\pm})$ (resp. 0) to $\mathrm{ind}(u)$ when $x_j^{\pm}$ is a negative (resp. positive) asymptote. The reversing of the roles of positive and negative asymptotes between here and the proof of Lemma 3.8 can be understood from the fact that in (3.19), positive asymptotes contribute positively while in (3.26), positive asymptotes contribute negatively, which in turn boils down to the reversing convention of the definition of indices between orbits and chords.

After all, we have

$$\mathrm{ind}(u) = n(1-r^-) + \sum_{j=1}^{r^-}(\iota(x_j^-) + \mathrm{mb}(x_j^-)) - \sum_{j=1}^{r^+}\iota(x_j^+) \tag{3.27}$$

The last term of (3.25) comes from the dimension of $\mathcal{R}^{d+1}$. □

Again, Lemma 3.10 applies also in the case when $Y^- = \varnothing$ or $Y^+ = \varnothing$.

Example 3.11 Let $q_0, q_1 \in S^n$ and Λ_i be the unit cospheres at q_i, and assume $T^*_{q_i}$ and the zero section are equipped with the canonical relative grading. Let x be the shortest Reeb chord from Λ_0 to Λ_1 in the unit cotangent bundle of S^n. The virtual dimension of $u : S \to T^*S^n$ such that $S \in \mathcal{R}^3$, $u(\partial_0 S) \subset S^n$, $u(\partial_1 S) \subset T^*_{q_0}S^n$, $u(\partial_2 S) \subset T^*_{q_1}S^n$ with positive asymptotes q_0 and x at ξ_1 and ξ_2, respectively, and a negative asymptote q_1 at ξ_0 is given by

$$\mathrm{virdim}(u) = n(1-1) + 0 - 0 - 0 = 0 \tag{3.28}$$

Finally, note that the shifting on the gradings of $T^*_{q_i}S^n$ or S^n do not change this virtual dimension (see Sect. 3.2).

Now, we combine Lemmas 3.8 and 3.10.

Lemma 3.12 *Let S be a disk with $r^+ + r^-$ boundary punctures and $s^+ + s^-$ interior punctures. Let $u : S \to X$ be a J-holomorphic map with positive asymptotes $\{x_j^+\}_{j=1}^{r^+}$ and negative asymptotes $\{x_j^-\}_{j=1}^{r^-}$ at boundary punctures, and positive asymptotes $\{\gamma_j^+\}_{j=1}^{s+}$ and negative asymptotes $\{\gamma_j^-\}_{j=1}^{s-}$ at interior punctures such that $u(\partial S)$ lies in the corresponding Lagrangians determined by the boundary asymptotes. Then the virtual dimension of u is given by*

$$\begin{aligned}\mathrm{virdim}(u) = (n-3)(1-s^+-s^-) &+ \sum_{j=1}^{s^+}\mu_{RS}(\gamma_j^+) \\ &- \sum_{j=1}^{s^-}\mu_{RS}(\gamma_j^-) + \frac{1}{2}\sum_{j=1}^{s^+}\dim(S_{\gamma_j^+}) \\ &+ \frac{1}{2}\sum_{j=1}^{s^-}\dim(S_{\gamma_j^-}) + 2c_1^{rel}(TX)([u]) \\ &+ \sum_{j=1}^{r^-}(\iota(x_j^-) + \mathrm{mb}(x_j^-)) - \sum_{j=1}^{r^+}\iota(x_j^+) - (n-1)r^- + r^+\end{aligned} \tag{3.29}$$

Proof We follow the proof in [2, Proposition 11.13]. The domain S is the connected sum of a disk S_1 with $r^+ + r^-$ boundary punctures and a sphere S_2 with $s^+ + s^-$ interior punctures. Let $u_1 : S_1 \to X$ be a J-holomorphic map with positive asymptotes $\{x_j^+\}_{j=1}^{r^+}$ and negative asymptotes $\{x_j^-\}_{j=1}^{r^-}$ such that $u_1(\partial S_1)$ lies in the corresponding Lagrangians determined by the boundary asymptotes. Let $u_2 : S_2 \to X$ be a J-holomorphic map with positive asymptotes $\{\gamma_j^+\}_{j=1}^{s+}$ and negative asymptotes $\{\gamma_j^-\}_{j=1}^{s-}$. Then, we have

$$\mathrm{ind}(u) = \mathrm{ind}(u_1) + \mathrm{ind}(u_2) - 2n \tag{3.30}$$

which can be computed by (3.22) and (3.27). Finally, to get the virtual dimension, we need to add the dimension of the Teichmüller space that S lies, which is $(r^- + r^+ - 3) + 2(s^+ + s^-)$. It gives the formula (3.29). □

We want to use Lemmas 3.8 and 3.12 to derive some corollaries for the holomorphic buildings $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ obtained in Theorem 3.3. Let $v \neq v'$ be adjacent to the edge e. If $l_{\mathcal{T}}(v) + 1 = l_{\mathcal{T}}(v')$, then there is a Reeb chord x (or orbit γ) which is the positive asymptote of u_v at $f_v^{-1}(e)$ and the negative asymptote of $u_{v'}$ at $f_{v'}^{-1}(e)$. Let $u_v \#_x u_{v'}$ (resp $u_v \#_\gamma u_{v'}$) be a pseudo-holomorpic map with boundary and asymptotic conditions determined by gluing u_v and $u_{v'}$ along x (resp. γ). By a direct application of Lemmas 3.8 and 3.12, we get

$$\begin{cases} \mathrm{virdim}(u_v \#_x u_{v'}) = \mathrm{virdim}(u_v) + \mathrm{virdim}(u_{v'}) - \dim(S_x) \\ \mathrm{virdim}(u_v \#_\gamma u_{v'}) = \mathrm{virdim}(u_v) + \mathrm{virdim}(u_{v'}) - \dim(S_\gamma) \end{cases} \tag{3.31}$$

On the other hand, if $l_{\mathcal{T}}(v) = l_{\mathcal{T}}(v')$ so that there is a Lagrangian intersection point x which is the positive asymptote of u_v at $f_v^{-1}(e)$ and the negative asymptote of $u_{v'}$ at $f_{v'}^{-1}(e)$, then we have

$$\mathrm{virdim}(u_v \#_x u_{v'}) = \mathrm{virdim}(u_v) + \mathrm{virdim}(u_{v'}) + 1 \tag{3.32}$$

where $u_v \#_x u_{v'}$ is defined analogously.

3.4 Action

This subsection discuss the action of the generators. A similar discussion can be found in [32] and [27, Section 3].

Let L_0, L_1 be exact Lagrangians in (M, ω, θ). It means that, for $j = 0, 1$, there exists a primitive function $f_j \in C^\infty(L_j, \mathbb{R})$ such that $df_j = \theta|_{L_j}$. For a Lagrangian intersection point $p \in CF(L_0, L_1)$, the action is

$$A(p) = f_0(p) - f_1(p) \tag{3.33}$$

so $A(p^\vee) = -A(p)$ (see Convention 3.7). For a contact hypersurface $(Y, \alpha = \theta|_Y) \subset (M, \omega, \theta)$ and a Reeb chord $x : [0, l_x] \to Y$ from $\Lambda_0 = L_0 \cap Y$ to $\Lambda_1 = L_1 \cap Y$. The length of x is

$$L(x) = \int x^* \alpha = l_x \tag{3.34}$$

and the action is

$$A(x) = L(x) + (f_0(x(0)) - f_1(x(l_x))) \tag{3.35}$$

Reeb orbits are special kinds of Reeb chords so the length and action of a Reeb orbit γ is

$$L(\gamma) = A(\gamma) = \int \gamma^*\alpha \tag{3.36}$$

We have the following action control.

Lemma 3.13 see [27](Lemma 3.3, Proposition 3.5) *Let $u_\infty = \{u_v\}_{v\in V(\mathcal{T})}$ be a holomorphic building obtained in Theorem 3.3. If u_v has positive asymptotes $\{x_j^+\}_{j=1}^{r^+}$, $\{\gamma_j^+\}_{j=1}^{s^+}$ and negative asymptotes $\{x_j^-\}_{j=1}^{r^-}$, $\{\gamma_j^-\}_{j=1}^{s^-}$, then*

$$E_\omega(u_v) := \sum_{j=1}^{r^+} A(x_j^+) + \sum_{j=1}^{s^+} A(\gamma_j^+) - \sum_{j=1}^{r^-} A(x_j^-) - \sum_{j=1}^{s^-} A(\gamma_j^-) \geqslant 0 \tag{3.37}$$

The equality holds if and only if u_v is a trivial cylinder [*see* (3.3)].

Since $A(\gamma) > 0$ for any Reeb orbit γ and $A(x) > 0$ if x is a non-constant Reeb chord such that $x(0) = x(l_x)$, a direct consequence of Lemma 3.13 is

Corollary 3.14 *If $u_v : \Sigma_v \to SM^+$ has only negative asymptotes, then at least one of the asymptotes is not a Reeb orbit nor a Reeb chord x such that $x(0) = x(l_x)$.*

Lemma 3.15 *Let $u_\infty = \{u_v\}_{v\in V(\mathcal{T})}$ be a holomorphic building obtained in Theorem 3.3. Let x_j be the boundary punctures corresponding to the leaves and root edges of $\mathcal{T}$ If $\sum_{j=0}^{d} |A(x_j)| < T$, then for every $v \in V(\mathcal{T})$, the action of every asymptote of u_v lies in $[-T, T]$.*

Proof Let us assume the contrary. Then there is an asymptote of u_v with action lying outside $[-T, T]$. We assume that this is a boundary asymptote and denote it by x. The case for interior asymptote is identical. If $A(x) > T$ (resp. $A(x) < -T$), we pick $v' \in V(\mathcal{T})$ (which might be v itself) such that x is a negative (resp. positive) asymptote of $u_{v'}$. Let e be the edge in $\mathcal{T}$ corresponds to this asymptote. Let G be the subtree of $\mathcal{T}\backslash\{e\}$ containing v'.

Denote $x_{v,i}$ by the asymptotes corresponding to the vertex v. Let $sgn(x) = 0$ (resp. $sgn(x) = 1$) if x is a positive (resp. negative) asymptote. Then

$$\begin{aligned} 0 &\leqslant \sum_{v\in G} E_\omega(u_v) \\ &= \sum_{v\in G}\sum_{i}(-1)^{sgn(x_{i,v})} A(x_{i,v}) \\ &\leqslant (-1)^{sgn(x)} A(x) + \sum_{j} |A(x_j)| < 0 \end{aligned} \tag{3.38}$$

Here $x_{i,v}$ runs over all asymptotes of u_v, and j over all semi-infinite edges. The second inequality holds because all finite edges are cancelled between the components they connect. This concludes the lemma. ◻

Lemma 3.16 *Let $u_\infty = \{u_v\}_{v \in V(\mathcal{T})}$ be a holomorphic building obtained in Theorem 3.3. If $v \in V^\partial \cup V^{int}$, then only the action of the asymptote of u_v that corresponds to the edge e_v closest to the root of $\mathcal{T}$ contributes positively to $E_\omega(u_v)$.*

Proof Let G_v be the subtree of $\mathcal{T} \backslash \{e_v\}$ containing v. We apply induction on the number of vertices in G_v.

If G_v has only one vertex, then $0 < E_\omega(u_v)$ is only contributed by the asymptote corresponds to e_v so the base case is done.

Now we consider the general case. Let e be an edge in G_v (so $e \neq e_v$). Let $v' \neq v$ be the other vertex adjacent to e so $v' \in V^\partial \cup V^{int}$ by Lemma 3.5. By induction on $G_{v'}$, we know that the asymptote corresponding to e contributes positively to $E_\omega(u_{v'})$ and hence negatively to $E_\omega(u_v)$. Finally, for $E_\omega(u_v)$ to be non-negative, we need to have at least one term which contributes positively to $E_\omega(u_v)$. This can only be contributed by the asymptote corresponding to e_v. □

Lemma 3.17 (Distinguished asymptote) *Let $u_\infty = \{u_v\}_{v \in V(\mathcal{T})}$ be a holomorphic building obtained in Theorem 3.3. If $\partial \Sigma_v \neq \emptyset$ and u_v is not a trivial cylinder* [*see* (3.3)], *then there is a boundary asymptote x of u_v that appears only once among all the asymptotes $\{x_i^\pm\}$ of u_v.*

Proof By Lemma 3.16, when $v \in V^\partial$, the asymptote of u_v at ξ_0^v is the only asymptote that contributes positively to energy and hence appears only once among the asymptotes of u_v.

Now, we consider $v \in V^{core}$. If there are more than two Lagrangians appearing in the Lagrangian labels of $\partial \Sigma_v$, say, $\partial_j S$ and $\partial_{j+1} S$ are labelled by L_{k_1} and L_{k_2}, respectively, for $k_1 \neq k_2$, then the asymptote of u_v at ξ_{j+1}^v can only appear once among the asymptotes of u_v, by Lagrangian boundary condition reason.

If there are exactly two Lagrangians appearing in the Lagrangian labels of $\partial \Sigma_v$, then there are exactly two j such that the Lagrangian labels on $\partial_j S$ and $\partial_{j+1} S$ are different. Let the two j be j_1 and j_2. It is clear that $f_v(\xi_{j_1+1}^v)$ and $f_v(\xi_{j_2+1}^v)$ are the only two edges in $\mathcal{T}^{core} \backslash (\mathcal{T}^\partial \cup \mathcal{T}^{int})$ that are adjacent to v. Therefore, by our first observation, the action of the asymptotes corresponding the other edges of v contributes negatively to $E_\omega(u_v)$.

If u_v converges to the same Reeb chord at $\xi_{j_1+1}^v$ and $\xi_{j_2+1}^v$, then one of it must be a positive asymptote and the other is a negative asymptote by Lagrangian boundary condition. Therefore, the contribution to $E_\omega(u_v)$ by this same asymptote cancels.

Similarly, if u_v converges to the same Lagrangian intersection point at $\xi_{j_1+1}^v$ and $\xi_{j_2+1}^v$, then the contribution to $E_\omega(u_v)$ by this same asymptote cancels because of the order of the Lagrangian boundary condition. As a result, we have $E_\omega(u_v) \leqslant 0$ which happens only when u_v is a trivial cylinder [see (3.3)], by Lemma 3.13. □

Remark 3.18 Notice that, when u_v maps to SY, the sum (3.37) becomes

$$\sum_{j=1}^{r^+} L(x_j^+) + \sum_{j=1}^{s^+} L(\gamma_j^+) - \sum_{j=1}^{r^-} L(x_j^-) - \sum_{j=1}^{s^-} L(\gamma_j^-) \tag{3.39}$$

because the terms involving the primitive functions on the Lagrangians add up to zero.

3.5 Morsification

We come back to our focus on $U = T^*P$, where P satisfies (2.58). We will need to use a perturbation of the standard contact form α_0 on ∂U to achieve transversality later. In this section, we explain how the action and index of the Reeb chord/orbit are changed under such a perturbation.

As explained in Sect. 3.2, $(\partial U, \alpha_0)$ is foliated by Reeb orbits. The quotient of ∂U by the Reeb orbits is an orbifold, which is denoted by $Q_{\partial U}$. We can choose a Morse function $f_Q : Q_{\partial U} \to \mathbb{R}$ compatible with the strata of $Q_{\partial U}$ and lifts f_Q to a R_{α_0}-invariant function $f_\partial : \partial U \to \mathbb{R}$ (see [37, Section 2.2]). Let $\mathrm{critp}(f_Q)$ be the set of critical points of f_Q. Let $\alpha = (1 + \delta f_\partial)\alpha_0$, which is a contact form for $|\delta| \ll 1$. Let $L(\partial U)$ be the length of a generic simple Reeb orbit of $(\partial U, \alpha_0)$.

Lemma 3.19 ([37] Lemma 2.3) *For all $T > L(\partial U)$, there exists $\delta > 0$ such that every simple α-Reeb orbit γ with $L(\gamma) < T$ is non-degenerate and is a simple α_0-Reeb orbit. Moreover, the set of simple α-Reeb orbits γ with $L(\gamma) < T$ is in bijection to* $\mathrm{critp}(f_Q)$.

Furthermore, if γ is the m-fold cover of a simple α-Reeb orbit γ^s such that $L(\gamma) < T$, then

$$\mu^{\alpha_0}(\gamma) + \frac{1}{2}dim(S_\gamma) \geqslant \mu^{\alpha}_{RS}(\gamma) \geqslant \mu^{\alpha_0}_{RS}(\gamma) - \frac{1}{2}\dim(S_\gamma) \tag{3.40}$$

where $\mu^{\alpha}_{RS}(\gamma)$, $\mu^{\alpha_0}_{RS}(\gamma)$ are the Robbin–Salamon index of γ with respect to α and α_0, respectively, and S_γ is the Morse–Bott family with respect to α_0 that γ lies.

Proof The first statement follows from [37, Lemma 2.3].

For the second statement, we need to compare the path of symplectic matrices Φ^α_t, $\Phi^{\alpha_0}_t$ corresponding to α and α_0, respectively. We can isotope the Poincare return map $\Phi^{\alpha_0}_t$ relative to end points, by changing the trivialization, to $\widetilde{\Phi}^{\alpha_0}_t$ such that $\ker(\widetilde{\Phi}^{\alpha_0}_t - Id) \neq 0$ only happens at finitely many $t \in [0, 1]$, where all such t contribute transversely. For a fixed T, we can choose δ sufficiently small such that Φ^α_t and $\widetilde{\Phi}^{\alpha_0}_t$ are arbitrarily close but with $\ker(\widetilde{\Phi}^{\alpha}_t(1) - Id) \neq 0$. As a result, only the last contribution to $\mu^{\alpha_0}_{RS}(\gamma)$ at $t = 1$ may not persist [see (3.15)] and we obtain the result. □

Corollary 3.20 *For all $T > L(\partial U)$, there exists $\delta > 0$ such that every α-Reeb orbit γ with $L(\gamma) < T$ and being contractible in U has $\mu^\alpha_{RS}(\gamma) \geqslant n-1$. As a result, the virtual dimension of $u : \mathbb{C} \to SM^-$ with positive asymptote γ satisfies* $\mathrm{virdim}(u) \geqslant 2n - 4$.

Proof The underlying simple Reeb orbit γ^s of γ must have $L(\gamma^s) < T$ so it is also a α_0-Reeb orbit, by Lemma 3.19. Since γ is contractible in U, by the explanation in Sect. 3.2.4, we have $\mu^{\alpha_0}_{RS}(\gamma) = 2k(n-1)$ for some $k > 0$ and $\dim(S_\gamma) = 2n-2$. Therefore, $\mu^\alpha_{RS}(\gamma) \geqslant n - 1$ by Lemma 3.19 and $\mathrm{virdim}(u) = (n-3) + \mu^\alpha_{RS}(\gamma) \geqslant 2n - 4$. □

We have a similar index calculation for Reeb chords:

$$\iota(x_0) - dim(S_{x_0}) \leqslant \iota(x) \leqslant \iota(x_0) + dim(S_{x_0}). \tag{3.41}$$

The proof is identical to Lemma 3.19 hence omitted. Let $\Lambda_q \subset \partial U$ be the cosphere at q.

Lemma 3.21 *There exists f_Q such that for all $T > L(\partial U)$, there exists $\delta > 0$ such that every α-Reeb chord x from Λ_{q_1} to Λ_{q_2} with $L(x) < T$ has $|x| \leqslant 0$ in the canonical relative grading. Here, we allow $q_1 = q_2$.*

Moreover, if q_i are in relatively generic position on P, for each lift $\mathbf{q}_i$ of q_i, there is exactly one such chord $x_{\mathbf{q_1},\mathbf{q_2}}$ with $|x_{\mathbf{q_1},\mathbf{q_2}}| = 0$ in canonical relative grading such that $x_{\mathbf{q_1},\mathbf{q_2}}$ can be lifted to a Reeb chord from $\Lambda_{\mathbf{q}_1}$ to $\Lambda_{\mathbf{q}_2}$.

Proof For the first statement, when $\delta > 0$ is sufficiently small, x is C^1-close to a α_0-Reeb chord from Λ_q to itself. Recall from Sect. 3.2.3 that, a non-degenerate α_0-Reeb chord x_0 from Λ_q to itself has $\iota(x_0) \leqslant 0$. Therefore, if x is C^1-close to x_0, then $\iota(x) \leqslant 0$.

On the other hand, a degenerated α_0-Reeb chord x_0 from Λ_q to itself has $\iota(x_0) = -k(n-1) \leqslant -(n-1)$ for some $k > 0$. We have $\dim(S_{x_0}) = n-1$, so if x is C^1-close to x_0, then $\iota(x) \leqslant \iota(x_0) + \dim(S_{x_0}) \leqslant -(n-1) + (n-1) = 0$. The first inequality comes from (3.41).

For the second statement, we only need to notice that $|x_{\mathbf{q_1},\mathbf{q_2}}| = 0$ if an only if the chord is the lift of (a perturbation of) the unique geodesic between $\mathbf{q_1}$ and $\mathbf{q_2}$ with length less than π from (3.12). $\square$

Note that, we do not need to assume x is non-degenerate in Lemma 3.21.

After choosing α in Lemma 3.19, there are only finitely many simple Reeb orbits of length less than T. They correspond to finitely many geodesic loops in P. Therefore, for generic (on the complement of the geodesic loops) $q \in P$, Λ_q does not intersect with simple Reeb orbits of length less than T. Moreover, for generic perturbation of f_Q, we can achieve the following:

Lemma 3.22 *We assume $n \geqslant 2$. For generic C^2-small perturbation of f_Q away from* $\mathrm{critp}(f_Q)$ *(such that the set* $\mathrm{critp}(f_Q)$ *is unchanged), every α-Reeb chord x from Λ_q to itself with $L(x) < T$ satisfies $x(t) \notin \Lambda_q$ for $t \in (0, L(x))$. Moreover, we can assume every such x is non-degenerate.*

Proof Mike Usher has pointed out the following proof to the authors. Assume a chord x has interior insection $x(t_i)$, $i = 1, \ldots, k$ with Λ_q, then we may now choose a contactomorphism τ with small C^2-norm supported near $x(t_i)$, which pushes $x(t_i)$ off Λ_q for all i, and consider the contact form $\tau_* \alpha$. Since we did not change the contact structure, Λ_q remains Legendrian and the perturbation on the contact form is by a function f supported near $x(t_i)$. $\tau(x(t))$ is then a Reeb chord with no interior intersection with Λ_q, and from the transversality assumption and argument above, there is no new chords created. The induction on the number of chords concludes the lemma. $\square$

Corollary 3.23 *We assume $n \geqslant 2$. For all $T > L(\partial U)$ and $k \in \mathbb{N}$, there exists $\delta > 0$, $f_\partial : \partial U \to \mathbb{R}$ and pairwise distinct $q_1, \ldots, q_k \in P$ such that $\alpha = (1 + \delta f_\partial)\alpha_0$ satisfies*

(1) *every simple α-Reeb orbit γ that is contractible in U and $L(\gamma) < T$ is non-degenerate and $\mu_{RS}(\gamma) \geqslant n-1$, and*
(2) *every α-Reeb chord x from $\bigcup_{i=1}^k \Lambda_{q_i}$ to $\bigcup_{i=1}^k \Lambda_{q_i}$ with $L(x) < T$ is non-degenerate, satisfies $x(t) \notin \bigcup_{i=1}^k \Lambda_{q_i}$ for $t \in (0, L(x))$ and $|x| \leqslant 0$ with respect to canonical relative grading.*

As a consequence, the image of the α-Reeb chords x from $\bigcup_{i=1}^k \Lambda_{q_i}$ to $\bigcup_{i=1}^k \Lambda_{q_i}$ with $L(x) < T$ are pairwise disjoint, and they are disjoint from the image of simple α-Reeb orbits.

Proof After choosing δ, f_Q such that (1) is satisfied by Lemma 3.19 and Corollary 3.20, and we can apply Lemma 3.22 to $\bigcup_{i=1}^k \Lambda_{q_i}$. Since the perturbation is arbitrarily C^2-small, we have $|x| \leqslant 0$ by Lemmas 3.21 and (3.12). □

In the rest of the paper, we always choose a contact form α on ∂U such that Corollary 3.23 holds, we denote the set of simple α-Reeb orbit γ with $L(\gamma) < T$ by $\mathcal{X}_T^o$. Similarly, we denote the set of α-Reeb chord x from $\bigcup_{i=1}^k \Lambda_{q_i}$ to $\bigcup_{i=1}^k \Lambda_{q_i}$ with $L(x) < T$ by $\mathcal{X}_T^c$.

3.6 Regularity

In this section, we address the regularity of curves u_v in the holomorphic buildings obtained in Theorem 3.3 for $v \in V^{core} \cup V^{\partial}$. We adapt the techniques developed in [24–26] and [27]. We borrow the observation made in [24]: if there is an asymptote that only appears once among the boundary asymptotes of a pseudo-holomorphic curve as proved in Lemma 3.17, then one can achieve regularity by perturbing J near the asymptote.

The main difference of our situation is that, we do not work in a contact manifold that is a contactization of an exact symplectic manifold, hence we don't have a projection of holomorphic curve as in [24,25]. We remedy the situation by localizing to a neighborhood of the Reeb chord.

We first explain the space of almost complex structure we use. In what follows, we always assume that a contact form α on ∂U is chosen such that Corollary 3.23 is satisfied.

Lemma 3.24 (Neighborhood theorem) *For any Reeb chord $x \in \mathcal{X}_T^c$, there exists a neighborhood N_x of $Im(x)$, an open ball $B_x \subset \mathbb{R}^{2n-2}$ containing the origin, an open interval $I_x \subset \mathbb{R}$ and a diffeomorphism $\phi_{N_x} : N_x \to B_x \times I_x$ such that*

$$\begin{cases} \alpha = \phi_{N_x}^*(dz + \sum_{i=1}^{n-1} x_i dy_i) \\ \pi_{B_x}(\phi_{N_x}(x(t))) = 0 \end{cases} \tag{3.42}$$

where $(x_i, y_i) \in B_x$, $z \in I_x$ and $\pi_{B_x} : B_x \times I_x \to B_x$ is the projection to the first factor.

Proof It follows from a Moser's argument. We give a sketch following [39, Theorem 2.5.1]. Since $d\alpha$ is non-degenerate on $T_pY/T_pIm(x)$ for all $p \in Im(x)$, we can use exponential map with respect to an appropriate metric to find coordinates $(x_1, y_1, \dots, x_{n-1}, y_{n-1}, z)$ near $Im(x)$ such that $Im(x) = \{x_i = y_i = 0\}$ and on $TY|_{Im(x)}$,

$$\begin{cases} \alpha(\partial_z) = 1, \iota_{\partial_z} d\alpha = 0 \\ \partial_{x_i}, \partial_{y_i} \in \ker(\alpha), d\alpha = \sum_{i=1}^{n-1} dx_i \wedge dy_i \end{cases} \tag{3.43}$$

Let $\alpha_{\mathbb{R}^{2n-1},std} = dz + \sum_{i=1}^{n-1} x_i dy_i$ and $\alpha_t = (1-t)\alpha_{\mathbb{R}^{2n-1},std} + t\alpha$. It follows that on $TY|_{Im(x)}$,

$$\alpha_t = \alpha, \qquad d\alpha_t = d\alpha \quad \text{for all } t \tag{3.44}$$

In particular, α_t is a family of contact forms in a sufficiently small neighborhood of $Im(x)$. By Moser trick, there exists a vector field X_t near $Im(x)$ such that the flow ψ_t satisfies $\psi_t^* \alpha_t = \alpha_{\mathbb{R}^{2n-1},std}$ for all $t \in [0, 1]$ and $X_t(p) = 0$ for all $p \in Im(x)$. We set $\phi_{N_x} = (\psi_1)^{-1}$. □

Remark 3.25 If we replace $dz + \sum_{i=1}^{n-1} x_i dy_i$ by $dz + \sum_{i=1}^{n-1} x_i dy_i + dy_1$ in Lemma 3.24, the lemma still holds.

Corollary 3.26 *Let B_x be one chosen in Lemma 3.24 or Remark 3.25. If J' is a compatible almost complex structure on B_x, then there is a cylindrical almost complex structure J on the symplectization $\mathbb{R} \times N_x$ such that $(\pi_{B_x} \circ \pi_Y)_* \circ J(v) = J' \circ (\pi_{B_x} \circ \pi_Y)_*(v)$ for all $v \in \xi$.*

Proof We can use the symplectic decomposition $T_{(r,z)}(\mathbb{R} \times N_x) = \mathbb{R}\langle \partial_r, R_\alpha \rangle \oplus \xi_z$ and the isomorphism $(\pi_{B_x})_* : \xi_z \simeq T_{\pi_{B_z}(z)} B_x$ to define J such that $J(\partial_r) = R_\alpha$ and $J(v) = ((\pi_{B_x} \circ \pi_Y)_*)^{-1} \circ J' \circ (\pi_{B_x} \circ \pi_Y)_*(v)$ for $v \in \xi_z$. One can check that J is a cylindrical almost complex structure. □

Now we may address the regularity of neck-stretching limits along ∂U. We summarize various auxiliary data chosen so far.

(1) Let $Y \subset (M, \omega, \theta)$ be a perturbation of ∂U such that $(Y, \theta|_Y) \cong (\partial U, \alpha)$. By abuse of notation, we denote $\theta|_Y$ by α.
(2) For the T chosen in Corollary 3.23, there are finitely many Reeb orbits or Reeb chord from $\bigcup_j \Lambda_{q_j}$ to $\bigcup_j \Lambda_{q_j}$ with length less than T. Moreover, the simple Reeb orbits $\mathcal{X}_T^o$ and the Reeb chords $\mathcal{X}_T^c$ have pairwise disjoint images.
(3) For each $x \in \mathcal{X}_T^c$, we pick a neighborhood N_x of $Im(x)$ using Remark 3.25. We assume that all these neighborhoods are pairwise disjoint and disjoint from the Reeb orbits of α.
(4) Let $x \in \mathcal{X}_T^c$, $x(0) \in \Lambda_{q_0}$ and $x(L(x)) \in \Lambda_{q_1}$. By Corollary 3.23, for sufficiently small N_x, we can assume that

$$D_{i,x} := \Lambda_{q_i} \cap N_x \tag{3.45}$$

is a disk for $i = 0, 1$. Moreover, by the fact that x is non-degenerate, we know that $\pi_{B_x}(D_{0,x})$ and $\pi_{B_x}(D_{1,x})$ are transversally intersecting Lagrangians. There exists a compatible J_{B_x} on B_x such that J_{B_x} is integrable near the origin. By possibly perturbing $\Lambda_{q,i}$, or equivalently perturbing α, we can assume that $\pi_{B_x}(D_{i,x})$ are real analytic submanifolds near origin for all x. We fix a choice of J_{B_x} for each $x \in \mathcal{X}_T^c$.

(5) Let $\mathcal{J}^{cyl}(\partial U; \{N_x\}_{x \in \mathcal{X}_T^c})$ be the space of $J \in \mathcal{J}^{cyl}(\partial U)$ such that J is R_α-invariant in N_x and there is a compatible almost complex structures J' on B_x so that $J' = J_{B_x}$ near the origin and $(\pi_{B_x} \circ \pi_Y)_* \circ J(v) = J' \circ (\pi_{B_x} \circ \pi_Y)_*(v)$ for all $v \in \xi$. By Corollary 3.26, we know that $\mathcal{J}^{cyl}(\partial U; \{N_x\}_{x \in \mathcal{X}_T^c}) \neq \varnothing$.

(6) We define $N(Y)$ as in (3.1). We can pick J^0 such that $(\Phi_{N(Y)})_* J^0|_{N(Y)} \in \mathcal{J}^{cyl}(\partial U; \{N_x\}_{x \in \mathcal{X}_T^c})$. Let $\{J^\tau\}_{\tau \in [3R, \infty)}$ be a smooth family R-adjusted to (Y, α) as explained in Sect. 3 (see Remark 3.2).

(7) We define $N(Y)$ as in (3.1). We can pick J^0 Let $\{L_j\}_{j=0}^d$ be a collection of Lagrangians satisfying the assumptions of Theorem 3.3. Moreover, we assume that $\Lambda_j = \bigcup_{i=1}^{c_j} \Lambda_{q_{k_{j,i}}}$ for some $q_{k_{j,i}}$ in Corollary 3.23. If T was chosen sufficiently large, there exists $0 < T^{adj} < T$ (depending only on the primitives of $\{L_j\}$, see Sect. 3.4) such that

$$\text{for all Reeb chords } x \text{ from } \Lambda_i \text{ to } \Lambda_j,\ |A(x)| < T^{adj} \text{ implies } |L(x)| < T \tag{3.46}$$

Without loss of generality, we can assume T^{adj} exists and $\sum_{j=0}^d |A(x_j)| < T^{adj}$. Applying Theorem 3.3 and Lemma 3.15, we get a holomorphic building $u_\infty = \{u_v\}_{v \in V(\mathcal{T})}$ such that all the asymptotes of u_v are either Lagrangian intersection points, Reeb chords in $\mathcal{X}_T^c$ or multiple cover of Reeb orbits in $\mathcal{X}_T^o$.

For u_∞, we have the following regularity result.

Proposition 3.27 (Regularity for intermediate level components) *There is a residual set $\mathcal{J}^{cyl,reg} \subset \mathcal{J}^{cyl}(\partial U; \{N_x\}_{x \in \mathcal{X}_T^c})$ such that if (the cylindrical extension of) $(\Phi_{N(Y)})_* J^0|_{N(Y)}$ lies in $\mathcal{J}^{cyl,reg}$, then for $v \in V^{core} \cup V^\partial$ and $l_\mathcal{T}(v) \in \{1, \dots, n_\mathcal{T} - 1\}$, the J^Y-holomorphic curve u_v is transversally cut out.*

Proof By Lemma 3.17, u_v has a boundary asymptote x that appears only once among its asymptotes. We want to show that transversality can be achieved by considering variation of almost complex structures in $SN_x := \mathbb{R} \times N_x$.

Let $\Lambda^{tot} = \bigcup_i \Lambda_{q_i}$ and $S\Lambda = \bigcup_i S\Lambda_{q_i}$ where Λ_{q_i} are obtained in Corollary 3.23. There is a Banach manifold $\mathcal{B}$ consisting of maps

$$u : (\Sigma_v, \partial \Sigma_v) \to (SY, S\Lambda^{tot}) \tag{3.47}$$

in an appropriate Sobolev class with positive weight (see [26,40]). Let U_Δ be an appropriate Banach manifold that is dense inside $\mathcal{J}^{cyl}(\partial U; \{N_x\}_{x \in \mathcal{X}_T^c})$. The map

$$(u, J) \mapsto \overline{\partial}_J u \tag{3.48}$$

defines a section $\mathcal{F}$ of a bundle $\mathcal{E}^{0,1} \to \mathcal{B} \times U_\Delta$ with differential

$$D\mathcal{F}(u, J) : T_u\mathcal{B} \times T_J U_\Delta \to \mathcal{E}^{0,1}_u \tag{3.49}$$

$$(\eta, \mathbf{Y}) \mapsto D_u(\eta) + \mathbf{Y}(u) \circ du \circ j_{\Sigma_v} \tag{3.50}$$

where j_{Σ_v} is the complex structure on Σ_v. By a choice of metric, we identify

$$T_u\mathcal{B} \simeq \Gamma(u^*TSY, u|^*_{\partial\Sigma_v} S\Lambda^{tot}) \tag{3.51}$$

where the right-hand side is the completion of the space of smooth sections in u^*TSY, which takes value in $u|^*_{\partial\Sigma_v} S\Lambda^{tot}$ along the boundary, with respect to an appropriate Sobolev norm. On the other hand, we have $\mathcal{E}^{0,1}_u = \Omega^{0,1}(u^*TSY)$, where the right hand side is the completion of the space of smooth u^*TSY-valued $(0, 1)$-form with respect to an appropriate Sobolev norm. We want to argue $D\mathcal{F}(u, J)$ is surjective at (u, J) using that fact that there exists a boundary asymptote $x \in \mathcal{X}^c_T$ of u that appears only once among its asymptotes and $\overline{\partial}_J u = 0$.

Suppose not, then there exists $0 \neq l \in \mathcal{E}^{0,1}_u$ such that

$$\langle l, D\mathcal{F}(u, J)(\eta, \mathbf{Y})\rangle_{L^2,\Sigma_v} = 0 \tag{3.52}$$

for all $\eta \in T_u\mathcal{B}$ and $\mathbf{Y} \in T_J U_\Delta$. By unique continuation principle, it suffices to show that $l = 0$ on some non-discrete set of Σ_v to get a contradiction.

Let $\mathcal{R} = u^{-1}(N_x) \subset \Sigma_v$ and we will show that for η supported in $\mathcal{R}$ and $\mathbf{Y}$ supported in SN_x, it is sufficient to get $l|_\mathcal{R} = 0$. By Lemma 3.24, we can identify SN_x with $\mathbb{R}_r \times (B_x)_{x_i,y_i} \times (I_x)_z$. Let $\overline{u} = \pi_{B_x} \circ \pi_Y \circ u_v|_\mathcal{R}$. In the coordinates $((r, z), (\{x_i\}, \{y_i\}))$, we can write $l|_\mathcal{R} = (l_1, l_2)$. For $\eta = 0$ and $\mathbf{Y}$ supported in SN_x,[1] (3.52) implies

$$\langle l_2, \mathbf{Y}(\overline{u}) \circ d\overline{u} \circ j_{\Sigma_v}\rangle_{L^2,\mathcal{R}} = 0 \tag{3.53}$$

where $\mathbf{Y}$ is r, z-invariant in SN_x by the definition of $\mathcal{J}^{cyl}(\partial U; \{N_x\}_{x\in\mathcal{X}^c_T})$ so $\mathbf{Y}(\overline{u})$ is well-defined.

Lemma 3.28 *It follows from* (3.53) *that* $l_2 = 0$.

Assuming Lemma 3.28, it suffices to show that $l_1 = 0$. Similarly, l_1 admits the unique continuation property (see [26, p. 754]) so we only need to show that $l_1 = 0$ on some non-discrete set of $\mathcal{R}$. For $\mathbf{Y} = 0$ and η supported in $\mathcal{R}$, (3.52) becomes

$$\langle l_1, D(\pi_{r,z}) \circ D_u\eta\rangle_{L^2,\mathcal{R}} = 0 \tag{3.54}$$

where $\pi_{r,z} : SN_x \to \mathbb{R}_r \times (I_x)_z$ is the projection. Notice that $J|_{T(\mathbb{R}_r\times(I_x)_z)}$ is the standard complex structure, and η depends on the domain $\mathcal{R}$ rather than the target SN_x. Therefore, we can find an interior point p of $\mathcal{R}$ and construct η appropriately

[1] $\mathbf{Y}$ vanishes along ∂_r, ∂_z and takes values in $\partial_{x_i}, \partial_{y_i}$.

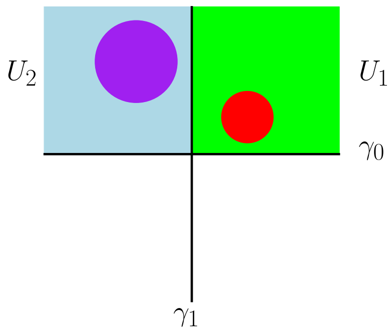

Fig. 2 Green region: U_1; Blue region: U_2; Red region: $B(x_1, \epsilon)$; Purple region: $B(x_2, \epsilon)$. Only $u_0(E_0)$ hits U_1 but not U_2 among $u_j(E_j)$ because unlike p_j (for $1 \leqslant j \leqslant s$), ξ_{j_x} is a boundary puncture (Color figure online)

supported near p to show that $l_1 = 0$. The details of the construction of η can be found in [26, page 754].

As a result, $l|_{\mathcal{R}} = 0$ and hence $l \equiv 0$. The existence of $\mathcal{J}^{cyl,reg}$ follows from applying Sard's-Smale theorem to the projection $\mathcal{F}^{-1}(0) \to U_\Delta$. □

Proof of Lemma 3.28 The proof is the same as [25, Lemma 4.5(1)]. For readers' convenience, we will recall the proof using our notation.

By the definition of $\mathcal{J}^{cyl}(\partial U; \{N_x\}_{x \in \mathcal{X}_T^c})$, $\overline{u}$ is a J'-holomorphic curve for some compatible almost complex structure J' on B_x such that $J' = J_{B_x}$ near origin. Moreover, exactly one boundary puncture, denoted by ξ_{j_x}, of $\mathcal{R}$ is mapped to the origin by our choice of x.

By the asymptotic behavior of holomorphic disks, we can assume that for sufficiently small $\delta > 0$, there exists a neighborhood $(E_0, \partial E_0) \subset (\mathcal{R}, \partial \mathcal{R})$ of ξ_{j_x} such that

(i) $(\overline{u}(E_0), \overline{u}(\partial E_0)) \subset (B(0, 2\delta), \pi_{B_x}(D_{0,x} \cup D_{1,x}) \cup \partial B(0, 2\delta))$,
(ii) $\pi_{B_x}(D_{0,x} \cup D_{1,x}) \cap \partial B(0, 2\delta)$ are two real analytic disjoint branches,
(iii) $\overline{u}(\partial E_0)$ contains two regular oriented curves $\gamma_0 \subset D_{0,x}$, $\gamma_1 \subset D_{1,x}$ in $B(0, 2\delta)$, respectively.

Here $B(0, 2\delta)$ is a 2δ-ball centered at the origin and $D_{i,x}$ are defined in (3.45).

To prove l_2 is zero we consider the variation of J' near a point on γ_0. To this end, we need to keep track of other parts of $\mathcal{R}$ that map onto γ_0.

Let $p_1 \dots, p_r \in \partial \mathcal{R}$ be the preimages under $\overline{u}$ of 0 with the property that one of the components of the punctured neighborhood of p_j in $\partial \mathcal{R}$ maps to γ_0. This set is finite and is identified with the set of boundary intersections between u and $\mathbb{R} \times x$.

Let $p_{r+1}, \dots p_s \in \mathcal{R} \backslash \partial \mathcal{R}$ be the preimages under $\overline{u}$ of 0 with the property that the preimage of γ_0 under $\overline{u}$ intersects some neighborhood of p_j in a 1-dimensional subset. By monotonicity lemma and maximum principle, this set is also finite, and is identified with the interior intersections between u and $\mathbb{R} \times x$.

For $1 \leqslant j \leqslant s$, let $E_j \subset \mathcal{R}$ denote the connected coordinate neighborhood of $\overline{u}^{-1}(B(0, 2\delta))$ near p_j. Let $U_1 = \overline{u}(E_0)$ and U_2 be the Schwartz reflection of U_1 through γ_1 (see Fig. 2).

By monotonicity lemma and maximum principle, for $i = 1, 2$ we can find a point $x_i \in U_i \backslash (B(0, \delta) \cup \pi_{B_x}(D_{0,x} \cup D_{1,x}))$ and small neighborhoods $B(x_i, \epsilon)$, $\epsilon \ll r$, such that

$$\overline{u}^{-1}(B(x_i, \epsilon)) \subset \bigcup_{j=0}^{s} E_j \tag{3.55}$$

When a certain branch of $\bar{u}(E_j) \not\pitchfork \bar{u}(E_0)$, then $x_i \bar{u}(E_j)$ for $i = 1, 2$ We exclude from our list any such $j \geqslant 1$. Note that for $j \geqslant 1, x_1 \in \overline{u}(E_j)$ if and only if $x_2 \in \overline{u}(E_j)$.

To simplify notation, we continue to index this possibly shortened list by $1 \leqslant j \leqslant s$.

For $1 \leqslant j \leqslant r$, we double the domain E_j through its real analytic boundary ∂E_j. We also double the local map $\overline{u}|_{E_j}$. We continue to denote the open disk by E_j. For $0 \leqslant j \leqslant s$, let $u_j = \overline{u}|_{E_j}$. We can also double (for $1 \leqslant j \leqslant r$) the cokernel element l_2 (which is anti-holomorphic) locally and define (for $0 \leqslant j \leqslant s$) $(l_2)_j = l_2|_{E_j}$.

There exists a disk $E \subset \mathbb{C}$ and a map f_E defined on E such that for $1 \leqslant j \leqslant s$, there exists positive integers k_j and bi-holomorphic identifications ϕ_j of E with E_j such that $(l_2)_j(\phi_j(z)) = f_E(z^{k_j})$ for $z \in E$.

Via our choice of perturbation of the complex structure, we can choose $\mathbf{Y}$ to be supported in $B(x_2, \epsilon)$. We get

$$\left\langle \sum_{j=1}^{s} (l_2)_j(\phi_j(z)), \mathbf{Y}(u_j \circ \phi_j) \circ d(u_j \circ \phi_j) \circ j_E \right\rangle_{L^2, E} = 0 \tag{3.56}$$

where j_E is the complex structure on E. Varying $\mathbf{Y}$, this implies

$$\sum_{j=1}^{s} (l_2)_j(\phi_j(z)) = 0 \tag{3.57}$$

We can also choose $\mathbf{Y}$ to be supported in $B(x_1, \epsilon)$. We get

$$\left\langle \sum_{j=1}^{s} (l_2)_j(\phi_j(z)), \mathbf{Y}(u_j \circ \phi_j) \circ d(u_j \circ \phi_j) \circ j_E \right\rangle_{L^2, E} + \langle (l_2)_0(z), \mathbf{Y}(u_0) \circ du_0 \circ j_{E_0} \rangle_{L^2, E_0} = 0$$

Since the first term is 0 by (3.57), by varying $\mathbf{Y}$, it implies $l_2|_{E_0} = (l_2)_0 = 0$ and hence $l_2 \equiv 0$. □

Next, we need to address the regularity when u_v lies in the top/bottom level of u_∞. We will explain the case that $l_\Upsilon(v) = n_\Upsilon$ (i.e. top level) in details and the other case is similar.

Let J_{M^+} be a compatible almost complex structure of SM^+ such that it is integrable near $SL_i^+ \pitchfork SL_j^+$, $i \neq j$. We assume that SL_i^+, SL_j^+ are real analytic near $SL_i^+ \pitchfork SL_j^+$.

For $J^Y \in \mathcal{J}^{cyl}(Y, \alpha)$, we let $\mathcal{J}^+(SM^+)$ to be the set of compatible almost complex structure J such that $J = J_{M^+}$ near $\bigcup_{i \neq j} SL_i^+ \cap SL_j^+$ and there exists $R > 0$ so that $J^+|_{(-\infty, -R] \times \partial M^+} = J^Y|_{(-\infty, -R] \times Y}$.

Proposition 3.29 (Regularity for M^+-components) *There is a residual set $\mathcal{J}^{+,reg} \subset \mathcal{J}^+(SM^+)$ such that if $J^+ \in \mathcal{J}^{+,reg}$, then for $v \in V^{core} \cup V^\partial$ and $l_\Upsilon(v) = n_\Upsilon$, the J^+-holomorphic curve u_v is transversally cut out.*

Proof By Lemma 3.17, u_v has a boundary asymptote x that appears only once among its asymptotes. If the distinguished asymptote of u_v is a Lagrangian intersection point, then we can apply the argument in [25, Lemma 4.5(1)] or Lemma 3.28 again to achieve the regularity of u_v. If the distinguished asymptote of u_v is a Reeb chord, we denote the corresponding puncture by $\xi^v_{j_x}$. By the asymptotic behavior of u_v, for a sufficiently large R, the preimage of a small neighborhood of $(-\infty, -R] \times Im(x)$ under u_v is a neighborhood of $\xi^v_{j_x}$. Therefore, we can find a somewhere injectivity point near $\xi^v_{j_x}$. Similar to the situation in SY, we can perturb J in SM^+ as long as J is cylindrical outside a compact set. Therefore, we can use the somewhere injectivity point to achieve regularity (see [27, Proposition 4.19] for exactly the same argument). □

Similarly, one define $\mathcal{J}^-(SM^-)$ analogously and we have

Proposition 3.30 *There is a residue set $\mathcal{J}^{-,reg} \subset \mathcal{J}^-(SM^-)$ such that if $J^- \in \mathcal{J}^{-,reg}$, then for $v \in V^{core} \cup V^\partial$ and $l_{\mathcal{T}}(v) = 0$, the J^--holomorphic curve u_v is transversally cut out.*

Remark 3.31 There is a possible alternative approach to the above regularity results if one could generalize the work of Lazzarini [41,42] and Perrier [43] to the SFT settings. This seems promising at least for $SM^\pm$, but the general regularity of SY might be more difficult.

3.7 No side bubbling

We can now summarize the previous discussion on u_∞ and draw geometric conclusions in this section.

Let L_j, $j = 0, \ldots, d$ be a collection of embedded exact Lagrangian submanifolds in (M, ω, θ) such that $L_i \pitchfork L_j$ for all $i \neq j$. Let P be a Lagrangian such that (2.58) is satisfied (P can be one of the L_j). Let U be a Weinstein neighborhood of P and we assume that $\theta|_U$ coincides with the canonical Liouville one form on T^*P. For $T \gg 1$, we pick α satisfying Corollary 3.23 and T^{adj} satisfying (3.46).

Let Y be a perturbation of ∂U such that $(Y, \theta|_Y) \cong (\partial U, \alpha)$. We denote $\theta|_Y$ by α. We have a neighborhood $\Phi_{N(Y)} : (N(Y), \omega|_{N(Y)}) \cong ((-\epsilon, \epsilon) \times Y, d(e^r\alpha))$ of Y. We assume that $L_j \cap N(Y) = (-\epsilon, \epsilon) \times \Lambda_j$ where $\Lambda_j = \bigsqcup \Lambda_{q_{jm}} = T^*_{q_{jm}} P \cap Y$ for some $q_{jm} \in P$ in Corollary 3.23.

Let J^τ be a smooth family of almost complex structures R-adjusted to $N(Y)$, such that $J^Y \in \mathcal{J}^{cyl,reg}$, where $\mathcal{J}^{cyl,reg}$ is obtained in Proposition 3.27. We also assume that $J^\pm \in \mathcal{J}^{\pm,reg}$, where $\mathcal{J}^{\pm,reg}$ is obtained in Propositions 3.29, 3.30.

Let $x_0 \in CF(L_0, L_d)$ and $x_j \in CF(L_{j-1}, L_j)$ for $j = 1, \ldots, d$. When T is large enough, we may assume that

$$\sum_{j=0}^{d} |A(x_j)| < T^{adj} \tag{3.58}$$

Suppose that there exists a sequence $\{\tau_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty}\tau_k = \infty$, and a sequence $u_k \in \mathcal{M}^{J^{\tau_k}}(x_0; x_d, \dots, x_1)$. We assume that $\mathrm{virdim}(u_k) = 0$. Let $u_\infty = \{u_v\}_{v\in V(\mathcal{T})}$ be the holomorphic building obtained in Theorem 3.3. Then we have

Proposition 3.32 (No side bubbling) *If $n \geqslant 3$, then $V^{int} = \varnothing$ and $n_{\mathcal{T}} = 1$. Moreover, if $v \in V^\partial$, then u_v is a rigid J^+-holomorphic map with exactly one boundary asymptote which is negative and goes to a Reeb chord.*

Proof For a subtree $G \subset \mathcal{T}$, we use $\mathrm{virdim}(G)$ to denote the virtual dimension of the map $\#_{v\in G}u_v$, where $\#_{v\in G}u_v$ refers to the map obtained by gluing all u_v such that $v \in G$ along the asymptotes determined by the edges. By (3.31), (3.32) and the fact that all Reeb chords/orbits arising as asymptotes of u_v are non-degenerate, we have

$$\mathrm{virdim}(G) = \sum_{v\in G}\mathrm{virdim}(u_v) + k_G \tag{3.59}$$

where k_G is the number of edges that correspond to Lagrangian intersections points and connect two distinct vertices in G. By assumption, $\mathrm{virdim}(\mathcal{T}) = 0$. Since u_v are transversally cut out for $v \in V^{core} \cup V^\partial$ (Propositions 3.27, 3.29, 3.30), we have $\mathrm{virdim}(u_v) \geqslant 0$. For $v \in V^{int}$, we cannot address the regularity but we have the following.

Lemma 3.33 *For each connected component G of $\mathcal{T}^{int}$, we have* $\mathrm{virdim}(G) > 0$.

Proof Let $v \in G$ be the vertex closest to the root. By 3.16, we have a distinguished interior puncture $\eta^0 \in \Sigma_v$ which contributes positively to $E_\alpha(u_v)$. Let γ^0 be the Reeb orbit that u_v is asymptotic to at η^0. Since $A(\gamma^0) = L(\gamma^0) > 0$, γ^0 must be a positive asymptote of u_v.

Notice that, by Corollary 3.14, there is no $v \in G$ such that u_v maps to SM^+. Therefore, $\#_{v\in G}u_v$ is a topological disk in SM^- so γ^0 is contractible in U. Moreover, $\mathrm{virdim}(G)$ is determined by γ^0 and it is given by $2n - 4 > 0$ (see Corollary 3.20). □

By combining (3.59), $\mathrm{virdim}(\mathcal{T}) = 0$, $\mathrm{virdim}(u_v) \geqslant 0$ for $v \in V^{core} \cup V^\partial$ and Lemma 3.33, we conclude that $V^{int} = \varnothing$, $k_G = 0$ and $\mathrm{virdim}(u_v) = 0$ for all v.

Notice that if u_v is not a trivial cylinder but $l_{\mathcal{T}}(v) \notin \{0, n_{\mathcal{T}}\}$, then $\mathrm{virdim}(u_v) \geqslant 1$ because one can translate u_v along the r-direction. Therefore, all intermediate level curves are trivial cylinders so $n_{\mathcal{T}} = 1$. The last thing to show is that if $v \in V^\partial$, then $l_{\mathcal{T}}(v) = 1$ and u_v has only one boundary asymptote.

We argue by contradiction. Suppose $l_{\mathcal{T}}(v) = 0$. Due to the boundary condition, all asymptotes of u_v are Reeb chords $y_0, \dots, y_{d_v}$. Inside SM^-, we can compute the index of Reeb chords using the canonical relative grading. By Corollary 3.23, we have $\iota(y_j) \leqslant 0$ for all j. It means that $\mathrm{virdim}(u_v) = n - \sum_{j=0}^{d_v}\iota(y_j) - (2-d_v) \geqslant n-2 > 0$. This is a contradiction so $l_{\mathcal{T}}(v) = 1$ for all $v \in V^\partial$.

Finally, if there exists $v \in V^\partial$ such that u_v has more than one boundary asymptote, then by the fact that $\mathcal{T}$ is a tree, we must have $v \in V^\partial$ such that $l_{\mathcal{T}}(v) = 0$. This is a contradiction so we finish the proof of Proposition 3.32. □

3.8 Gluings in SFT

To conclude our discussion on generalities of neck-stretching, we recall the following gluing theorem for SFT, which will play an important role in our proof.

Theorem 3.34 *Let $u_\infty = (u_v)_{v \in V(\mathcal{T})} \in \mathcal{M}^{J^\infty}(x_0; x_d, \ldots, x_1)$ be a holomorphic building such that u_v is transversally cut out for all v and* $\mathrm{virdim}(u_\infty) = 0$. *Assume also that all asymptotic Reeb chords are non-degenerate.*

Then for any small neighborhood N_{u_∞} of u_∞ in an appropriate topology, there exists $\Upsilon > 0$ sufficiently large such that for each $\tau > \Upsilon$, there is a unique $u^\tau \in \mathcal{M}^{J^\tau}(x_0; x_d, \ldots, x_1)$ lying inside N_{u_∞}. Moreover, u^τ is regular and $\{u^\tau\}_{\tau \in [\Upsilon, \infty)}$ converges in SFT sense to u_∞ as τ goes to infinity.

A nice account for the SFT gluing results can be found in "Appendix A" of [44]. In the presence of conical Lagrangian boundary conditions as in above, see also [25, Proposition 4.6] and [24, Section 8]. Theorem 3.34 is essentially the same as Proposition 4.6 in [25], except our contact manifold is not $P \times \mathbb{R}$. But this is not a concern for the gluing argument because the argument involves local analysis on a neighborhood of the holomorphic building, which is not affected by the global topology.

The typical application of Proposition 3.32 and Theorem 3.34 goes as follows. Given a collection of Lagrangians such that the assumption of Theorem 3.3 is satisfied, we want to determine the signed count of rigid elements in $\mathcal{M}^{J^\tau}(x_0; x_d, \ldots, x_1)$ for some large τ. When $d = 1$ (resp. $d = 2$), the signed count is responsible to the Floer differential (resp. Floer multiplication). If we pick $u_k \in \mathcal{M}^{J^{\tau_k}}(x_0; x_d, \ldots, x_1)$ such that $\lim_{k \to \infty} \tau_k = \infty$, we get a holomorphic building u_∞ by Theorem 3.3. By Proposition 3.32, u_∞ satisfies the assumption of Theorem 3.34. Therefore, for sufficently large τ, $\mathcal{M}^{J^\tau}(x_0; x_d, \ldots, x_1)$ is in bijection to $\mathcal{M}^{J^\infty}(x_0; x_d, \ldots, x_1)$. Moreover, all elements in $\mathcal{M}^{J^\tau}(x_0; x_d, \ldots, x_1)$ are transversally cut out. It means that the Floer differential (resp. Floer multiplication) can be computed by determining $\mathcal{M}^{J^\infty}(x_0; x_d, \ldots, x_1)$, which is exactly what we will do in the following section.

4 Cohomological identification

Let P be a Lagrangian such that (2.58) is satisfied and $\mathcal{P}$ be the universal local system on P. We pick a parametrization of P so that τ_P can be defined. In this section, we want to prove that

Proposition 4.1 *For $\mathcal{E}^0, \mathcal{E}^1 \in Ob(\mathcal{F})$, we have cohomological level isomorphism*

$$H(hom_{\mathcal{F}^{\mathrm{perf}}}(\mathcal{E}^0, T_{\mathcal{P}}(\mathcal{E}^1))) \simeq H(hom_{\mathcal{F}}(\mathcal{E}^0, \tau_P(\mathcal{E}^1))) \tag{4.1}$$

We will only consider the case that $\mathcal{E}^i = L_i$ are Lagrangians without local system. The proof of the general case is identical except that the notations become more involved. In slightly more geometric terms, we would like to directly construct a chain map ι from

$$C_0 := Cone(CF(\mathcal{P}, L_1) \otimes_\Gamma CF(L_0, \mathcal{P}) \xrightarrow{ev_\Gamma} CF(L_0, L_1)) \tag{4.2}$$

to

$$C_1 := CF(L_0, \tau_P L_1) \tag{4.3}$$

which induces isomorphism on cohomology.

By applying a Hamiltonian perturbation, we assume $L_0 \pitchfork L_1$, and that each connected component of $L_i \cap U$ is a cotangent fiber in U. The cotangent fiber $T_q^* P \cap U$ has $|\Gamma|$ different lifts $\{T_{g\mathbf{q}}^* \mathbf{P}\}_{g \in \Gamma} \cap \mathbf{U}$ in $\mathbf{U}$, where $\mathbf{U} \subset T^*\mathbf{P}$ is the universal cover of U. We assume the Dehn twist τ_P is supported inside U and we have a commutative diagram:

$$\begin{array}{ccc} \mathbf{U} & \xrightarrow{\tau_{\mathbf{P}}} & \mathbf{U} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{\tau_P} & U \end{array}$$

where $\pi : \mathbf{U} \to U$ is the covering map. As always, we assume that L_0, L_1 are equipped with $\mathbb{Z}-$gradings and spin structures.

Our strategy is to study directly the Floer cochain complexes from both sides of (4.1). Section 4.1 gives a geometric correspondence between the generators from the two sides, and Sect. 4.3 will study the SFT limits of involved holomorphic strips and triangles. Section 4.4 use a local model to compute several key contribution of moduli spaces in the SFT limits, which eventually leads to the matching of differentials of (4.1) in Sect. 4.5. Due to the heaviness of notation and length of our proof, we also included a more technical guide in Sect. 4.2, in hope of keeping the readers on board.

4.1 Correspondence of intersections

We denote the set of generators in C_0 by $\mathcal{X}(C_0)$, which is divided into two types $\mathcal{X}_a(C_0)$ and $\mathcal{X}_b(C_0)$:

- $\mathcal{X}_a(C_0)$: generators in $hom(\mathcal{P}, L_1) \otimes_\Gamma hom(L_0, \mathcal{P})[1]$
- $\mathcal{X}_b(C_0)$: generators in $hom(L_0, L_1)$

More precisely, $\mathcal{X}_b(C_0) = L_0 \cap L_1$ and $\mathcal{X}_a(C_0)$ is the set of elements of the form $[\mathbf{q}^\vee \otimes g\mathbf{p}] \sim [\mathbf{q}^\vee g \otimes \mathbf{p}] \sim [(g^{-1}\mathbf{q})^\vee \otimes \mathbf{p}]$, where we are using the correspondence (2.39) and (2.40). On the other hand, we denote $L_0 \cap \tau_P L_1$ by $\mathcal{X}(C_1)$ which is a set of generators for C_1.

Let $p \in L_0 \cap P$, $q \in L_1 \cap P$ and $\mathbf{p}, \mathbf{q} \in \mathbf{P}$ be a lift of p and q, respectively. We also introduce the following notation

$$\begin{aligned} &\mathbf{c}_{\mathbf{p},\mathbf{q}} : \text{ the unique intersection } T_{\mathbf{p}}^*\mathbf{P} \cap \tau_{\mathbf{P}}(T_{\mathbf{q}}^*\mathbf{P}) \\ &c_{\mathbf{p},\mathbf{q}} := \pi(\mathbf{c}_{\mathbf{p},\mathbf{q}}), \text{ which is an intersection of} L_0 \cap \tau_P L_1 \end{aligned} \tag{4.4}$$

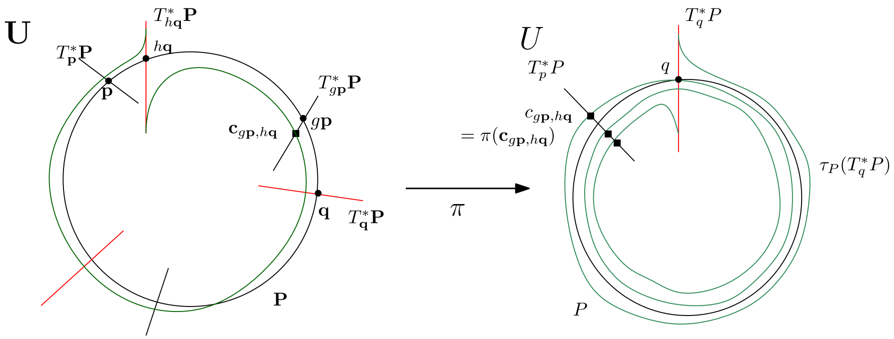

Fig. 3 Generator correspondence between C_0 and C_1

Lemma 4.2 *There is a grading-preserving bijection* $\iota : \mathcal{X}(C_0) \to \mathcal{X}(C_1)$.

Proof First, there is an obvious graded identification between $\mathcal{X}_b(C_0)$ and the intersections of $L_0 \cap \tau_P L_1$ outside U, so we only need to explain how to define $\iota|_{\mathcal{X}_a(C_0)}$.

We define $\iota|_{\mathcal{X}_a(C_0)}$ by

$$\iota|_{\mathcal{X}_a(C_0)} : \mathbf{q}^\vee \otimes \mathbf{p} \mapsto c_{\mathbf{p},\mathbf{q}} \tag{4.5}$$

This map is well-defined because

$$\iota(\mathbf{q}^\vee g^{-1} \otimes g\mathbf{p}) = \iota((g\mathbf{q})^\vee \otimes g\mathbf{p}) = \pi(\mathbf{c}_{g\mathbf{p},g\mathbf{q}}) = c_{\mathbf{p},\mathbf{q}} \tag{4.6}$$

The last equality comes from the equivariance of $\tau_{\mathbf{P}}$. It is clear that $\iota|_{\mathcal{X}_a(C_0)}$ is a bijection from $\mathcal{X}_a(C_0)$ to the intersections of $L_0 \cap \tau_P L_1$ inside U.

To see that $\iota|_{\mathcal{X}_a(C_0)}$ preserves the grading, we only need to observe that π intertwines the canonical trivialization of $(\Lambda_{\mathbb{C}}^{\otimes top}(T^*\mathbf{U}))^{\otimes 2}$ and $(\Lambda_{\mathbb{C}}^{\otimes top}(T^*U))^{\otimes 2}$ so the computation reduces to the case that $P = S^n$, which is well-known (see e.g. [1]). $\square$

Using Lemma 4.2, we define $\mathcal{X}_a(C_1) = \iota(\mathcal{X}_a(C_0))$ and $\mathcal{X}_b(C_1) = \iota(\mathcal{X}_b(C_0))$. We summarize our notation in Fig. 3.

4.2 Overall strategy

The differentials in C_0 can be divided into four types.

- Type (A1): differentials in $hom(L_0, \mathcal{P})$, i.e. pseudo-holomorphic strips in $\mathcal{M}(\mathbf{p}'; \mathbf{p})$
- Type (A2): differentials in $hom(\mathcal{P}, L_1)$, i.e. pseudo-holomorphic strips in $\mathcal{M}((\mathbf{q}')^\vee; \mathbf{q}^\vee)$
- Type (B): differentials in $hom(L_0, L_1)$, i.e. pseudo-holomorphic strips in $\mathcal{M}(x_0; x_1)$
- Type (C): differentials from the evaluation map, i.e. pseudo-holomorphic triangles in $\mathcal{M}(x; \mathbf{q}^\vee, \mathbf{p})$

For C_1, we divide the differentials similarly, using correspondence of generators ι. Concretely, we have:

- Type (A1′): pseudo-holomorphic strips in $\mathcal{M}(c_{\mathbf{p}',\mathbf{q}}; c_{\mathbf{p},\mathbf{q}})$;

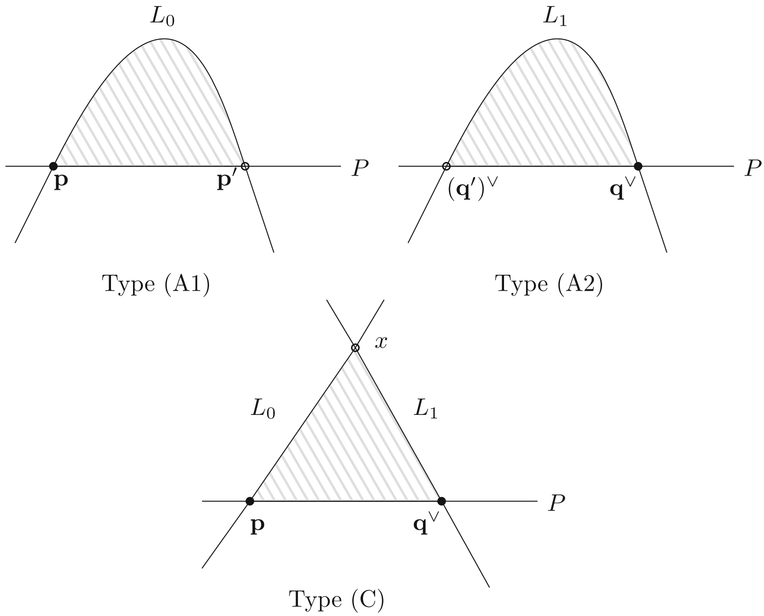

Fig. 4 Types of holomorphic curves in C_0

- Type (A2′): pseudo-holomorphic strips in $\mathcal{M}(c_{\mathbf{p},\mathbf{q}'}; c_{\mathbf{p},\mathbf{q}})$;
- Type (A3′): pseudo-holomorphic strips in $\mathcal{M}(c_{\mathbf{p}',\mathbf{q}'}; c_{\mathbf{p},\mathbf{q}})$ that are not in Type(A1′) and (A2′);
- Type (B′): pseudo-holomorphic strips in $\mathcal{M}(x_0; x_1)$;
- Type (C′): pseudo-holomorphic strips in $\mathcal{M}(x; c_{\mathbf{p},\mathbf{q}})$;
- Type (D′): pseudo-holomorphic strips in $\mathcal{M}(c_{\mathbf{p},\mathbf{q}}; x)$;

where $x, x_0, x_1 \in \mathcal{X}_b(C_1)$.

By the discussion in Sect. 3.8, we know that for an appropriate choice of $\{J^\tau\}$ and $\tau \gg 1$, all the rigid J^τ-holomorphic polygons in the moduli above are transversally cut out and they are bijective to the corresponding holomorphic buildings (Figs. 4 and 5). By studying the holomorphic buildings, we will show that there are bijective correspondences

$$\mathcal{M}^{J^\tau}(\mathbf{p}'; \mathbf{p}) \simeq \mathcal{M}^{J^\tau}(c_{\mathbf{p}',\mathbf{q}}; c_{\mathbf{p},\mathbf{q}}) \text{ for all } \mathbf{q}; \tag{4.7}$$

$$\mathcal{M}^{J^\tau}((\mathbf{q}')^\vee; \mathbf{q}^\vee) \simeq \mathcal{M}^{J^\tau}(c_{\mathbf{p},\mathbf{q}'}; c_{\mathbf{p},\mathbf{q}}) \text{ for all } \mathbf{p}; \tag{4.8}$$

$$\mathcal{M}^{J^\tau}(x_0; x_1) \simeq \mathcal{M}^{J^\tau}(x_0; x_1); \tag{4.9}$$

$$\mathcal{M}^{J^\tau}(x; \mathbf{q}^\vee, \mathbf{p}) \simeq \mathcal{M}^{J^\tau}(x; c_{\mathbf{p},\mathbf{q}}); \tag{4.10}$$

$$\text{Type (A3}'\text{) and (D}'\text{) are empty with respect to } J^\tau. \tag{4.11}$$

where the two sides of (4.9) are with respect to boundary conditions (L_0, L_1) and $(L_0, \tau_P(L_1))$, respectively. In other words, for $\tau \gg 1$, $\iota : C_0 \to C_1$ is an isomorphism which clearly implies Proposition 4.1.

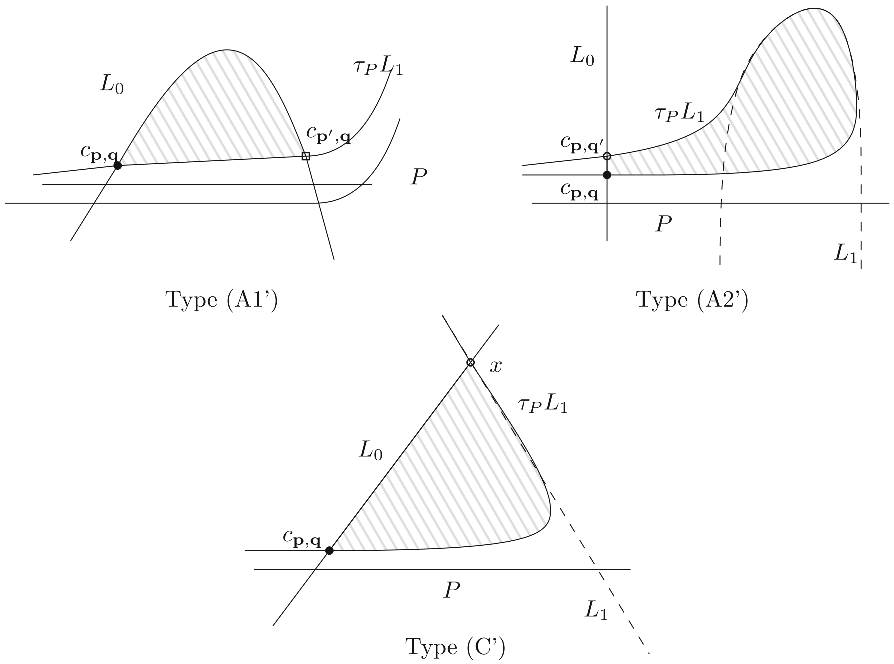

Fig. 5 Types of holomorphic curves in C_1

In the following subsections, we ignore the sign and only consider the case that $\mathrm{char}(\mathbb{K}) = 2$. The complete proof of Proposition 4.1, where orientation of moduli is taken into account, will be given in "Appendix A".

4.3 Neck-stretching limits of holomorphic strips and triangles

In this section, we will list all possible holomorphic buildings $u_\infty = \{u_v\}_{v\in V(\mathcal{T})}$ that arises as the limit (when $\tau \to \infty$) of curves in the moduli discussed in Sect. 4.2. By Proposition 3.32, we know that u_∞ satisfies the following conditions

$$\begin{aligned}
&(i) \text{ The total level } n_{\mathcal{T}} = 1,\\
&(ii) \ \mathrm{virdim}(u_v) = 0 \text{ for all } v.\\
&(iii) \text{ All compact edges in } \mathcal{T} \text{ correspond to Reeb chords.}\\
&(iv) \text{ If } v \in V^\partial, \text{ then } l_{\mathcal{T}}(v) = 1 \text{ and } u_v \text{ has exactly one boundary asymptote.}
\end{aligned} \tag{4.12}$$

Therefore, we assume (4.12) hold throughout this section. Recall also that Corollary 3.23 holds for our choice of $(\partial U, \alpha)$, hence the asymptotes under consideration are non-degenerate and $mb(x) = 1$ (cylindrical direction).

Lemma 4.3 *In the case (iv) of* (4.12)*, let $v \in V^\partial$ and x be the negative asymptote of u_v. Then $|x| = 1$.*

Proof By Lemma 3.10 and $\mathrm{virdim}(u_v) = 0$, we have

$$0 = \mathrm{virdim}(u_v) = |x| + \mathrm{mb}(x) - 2 = |x| - 1 \tag{4.13}$$

Therefore, $|x| = 1$. □

Lemma 4.4 *If $l_{\mathcal{T}}(v) = 0$, then u_v has at least one asymptote that is not a Reeb chord.*

Proof Suppose not. Let $y_1, \ldots, y_k$ be the asymptotes of u_v which are all positive Reeb chord. Notice that the shift of gradings for any individual boundary condition does not affect the virtual dimension of u_v. Therefore we can use the canonical relative grading to compute the virtual dimension of u_v. By Lemma 3.10 and Corollary 3.23, we have

$$\mathrm{virdim}(u_v) = n - \sum_{j=1}^{k} |y_j| - (3 - k) \geqslant n - 3 + k \geqslant n - 2 > 0 \tag{4.14}$$

which contradicts the assumption (4.12) that $\mathrm{virdim}(u_v) = 0$. □

Lemma 4.5 *Every generator $c_{\mathbf{p},\mathbf{q}} \in CF(T_p^* P, \tau_P(T_q^* P))$ satisfies $|c_{\mathbf{p},\mathbf{q}}| = n - 1$ with respect to the canonical relative grading. Moreover, if $c_{\mathbf{p},\mathbf{q}}$ is the only asymptote of a non-constant J^--holomorphic map $u_v : \Sigma_v \to SM^- = T^*P$ that is not a Reeb chord, then $c_{\mathbf{p},\mathbf{q}}$ must be positive as an asymptote of u_v.*

Proof To see that $|c_{\mathbf{p},\mathbf{q}}| = n - 1$, it suffices to show that $|\mathbf{c}_{\mathbf{p},\mathbf{q}}| = n - 1$. One can compute it directly by noting that $\tau_{\mathbf{P}}(T^*_{\mathbf{q}}\mathbf{P}) = \mathbf{P}[1]\#T^*_{\mathbf{q}}\mathbf{P}$, where $\mathbf{P}[1]$ is the grading shift of $\mathbf{P}$ by 1 and # denotes the graded Lagrangian surgery at the point $\mathbf{q}$ (see [29] or [21]). Alternatively, one can see it using the Dehn twist exact sequence [1]

$$0 \to HF^k(T^*_{\mathbf{p}}\mathbf{P}, \tau_{\mathbf{P}}(T^*_{\mathbf{q}}\mathbf{P})) \to \bigoplus_{a+b-1=k} HF^a(\mathbf{P}, T^*_{\mathbf{q}}\mathbf{P}) \otimes HF^b(T^*_{\mathbf{p}}\mathbf{P}, \mathbf{P}) \to 0 \tag{4.15}$$

and the fact that the second non-trivial term is non-zero only when $a = 0$ and $b = n$.

On the other hand, if $c_{\mathbf{p},\mathbf{q}}$ is a negative asymptote and the remaining asymptotes are denoted by $y_1, \ldots, y_k$, we would have (computed in canonical relative grading)

$$\mathrm{virdim}(u_v) = |c_{\mathbf{p},\mathbf{q}}| - \sum_{i=1}^{k} |y_i| - (2 - k) \geqslant n - 2 > 0 \tag{4.16}$$

which contradicts to the assumption (4.12) that $\mathrm{virdim}(u_v) = 0$. □

Now, we can describe the SFT limits of various moduli.

Lemma 4.6 (Type (A1)) *Let $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ be a non-empty SFT limit of curves in $\mathcal{M}^{J^\tau}(\mathbf{p}'; \mathbf{p})$. Then $\mathcal{T}$ consists of exactly two vertices v_1, v_2 and*

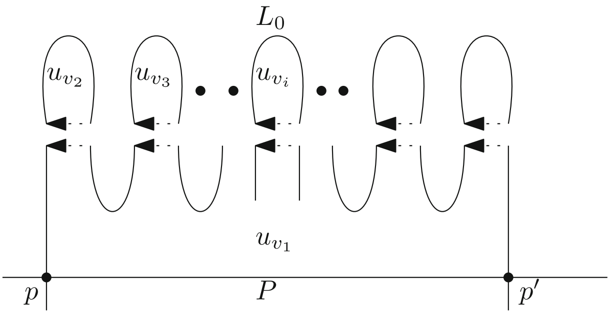

Fig. 6 Multiple side bubbles

- *u_{v_1} is a J^--holomorphic triangle with negative asymptote $p' := \pi(\mathbf{p}')$ and positive asymptotes x, p where x is a Reeb chord with $|x| = 0$ in the canonical relative grading;*
- *$v_2 \in V^\partial$ so, by Lemma 4.3, u_{v_2} is a J^+-holomorphic curve with one negative asymptote x such that $|x| = 1$ in the actual grading.*

Proof Notice that, by the boundary condition P, p and p' must be asymptotes of the same u_v. We call it u_{v_1}. We label the other vertices of $\mathcal{T}$ by $v_2, \dots, v_k$ for some $k \geqslant 0$. By boundary condition again, we know that $v_j \in V^\partial$ for $j > 1$. By (4.12), we have $l_{\mathcal{T}}(v_j) = 1$ for $j > 1$. Moreover, all v_j are adjacent to v_1 because u_{v_j} has a negative asymptote (see Fig. 6). By Lemma 3.10 and Corollary 3.23 again,

$$0 = \text{virdim}(u_{v_1}) = |p'| - |p| - \sum_{j=1}^{k} |y_j| - (1 - k) \geqslant k - 1 \tag{4.17}$$

so $k = 0, 1$. However, $k \neq 0$ by boundary condition. As a result, $k = 1$ and we denote y_1 by x.

Finally, to compute $|x|$ in the canonical relative grading, we just need to make a grading shift so that $|p'| - |p| = 1$ on $T^*_{p'}P$. It gives $|x| = 0$ in the canonical relative grading. □

Similarly, we have.

Lemma 4.7 (Type (A1$'$)) *Let $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ be a non-empty SFT limit of curves in $\mathcal{M}^{J^\tau}(c_{\mathbf{p}',\mathbf{q}}; c_{\mathbf{p},\mathbf{q}})$. Then $\mathcal{T}$ consists of exactly two vertices v_1, v_2 and*

- *u_{v_1} is a J^--holomorphic triangle with negative asymptote $c_{\mathbf{p}',\mathbf{q}}$ and positive asymptotes $x, c_{\mathbf{p},\mathbf{q}}$ where x is a Reeb chord with $|x| = 0$ in the canonical relative grading;*
- *$v_2 \in V^\partial$ so u_{v_2} is a J^+-holomorphic curve with one negative asymptote x such that $|x| = 1$ in the actual grading.*

We omit the corresponding statements for type (A2) and (A2$'$) because of the similarity. Next we consider

Lemma 4.8 (Type (B), (B′)) *Let $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ be a non-empty SFT limit of curves in $\mathcal{M}^{J^\tau}(x_0; x_1)$. Then $\mathcal{T}$ consists of exactly one vertex v and $l_{\mathcal{T}}(v) = 1$.*

Proof If $\mathcal{T}$ has a vertex v such that $l_{\mathcal{T}}(v) = 0$, then all the asymptotes of v are Reeb chords which contradicts to Lemma 4.4. Therefore, $l_{\mathcal{T}}(v) = 1$ for all $v \in V(\mathcal{T})$ and it holds only when $\mathcal{T}$ consists of exactly one vertex. □

Lemma 4.9 (Type (C)) *Let $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ be a non-empty SFT limit of curves in $\mathcal{M}^{J^\tau}(x; \mathbf{q}^\vee, \mathbf{p})$. Then $\mathcal{T}$ consists of exactly two vertices v_1, v_2 and*

- *u_{v_1} is a J^--holomorphic triangle with positive asymptotes $y, \mathbf{q}^\vee, \mathbf{p}$, where y is a Reeb chord with $|y| = 0$ in the canonical relative grading;*
- *u_{v_2} is a J^+-holomorphic curve with two negative asymptotes x and y.*

Proof Again, we use the same argument as in the proof of Lemma 4.6. There is $v_1 \in \mathcal{T}$ such that u_{v_1} is a holomorphic polygon and $\mathbf{q}^\vee$, $\mathbf{p}$ are asymptotes of u_{v_1}. All other vertices are adjacent to v_1: otherwise, there will be components in $T^*\mathbf{P}$ with only Reeb asymptotes, contradicting Lemma 4.4. Denote these vertices by $v_2, \ldots, v_k$. There is exactly one $j > 1$ (say $j = 2$) such that $v_j \notin V^\partial$ and x is an asymptote of u_{v_j}. For $\mathcal{T}$ to be a tree, u_{v_2} has exactly one negative Reeb chord asymptote, which is denoted by y_2.

Let the negative asymptote for u_{v_j} (for $j > 2$) be y_j.

For u_{v_1} to be rigid, we have

$$0 = n - |\mathbf{p}| - |\mathbf{q}^\vee| - \sum_{j=2}^{k} |y_j| - (2 - k) \geqslant n - n - 0 + k - 2$$

so $k \leqslant 2$. However, we have $k \geqslant 2$ so we get $k = 2$. Moreover, the canonical relative grading of y_2 is 0. □

Remark 4.10 Later on, we will also make use of the moduli space $\mathcal{M}^{J^\tau}(\mathbf{p}^\vee; x^\vee, \mathbf{q}^\vee)$. The shape of neck-stretching limit will remain the same as Type (C), because this is simply a modification of some of the strip-like ends (from outgoing to incoming, and vice versa) and does not change the behavior of the underlying curve.

Lemma 4.11 (Type (C′)) *Let $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ be a non-empty SFT limit of curves in $\mathcal{M}^{J^\tau}(x; c_{\mathbf{p},\mathbf{q}})$. Then $\mathcal{T}$ consists of exactly two vertices v_1, v_2 and*

- *u_{v_1} is a J^--holomorphic bigon with positive asymptotes $y, c_{\mathbf{p},\mathbf{q}}$, where y is a Reeb chord with $|y| = 0$ in the canonical relative grading;*
- *u_{v_2} is a J^+-holomorphic curve with two negative asymptotes x and y.*

Proof The argument is entirely parallel to Lemma 4.9. Let u_{v_1} be the J^--holomorphic curve such that $c_{\mathbf{p},\mathbf{q}}$ is an asymptote of it. Let the other asymptotes of u_{v_1} be $y_1, \ldots, y_k$. For u_{v_1} to be rigid, by Lemma 4.5,

$$0 = \mathrm{virdim}(u_{v_1}) = n - |c_{\mathbf{p},\mathbf{q}}| - \sum_{j=1}^{k} |y_j| - (2 - k) \geqslant n - (n - 1) - 2 + k = k - 1 \tag{4.18}$$

so $k = 1$ because u_{v_1} has at least one positive Reeb chord asymptote. □

Our final task is to show that type (A3′) and (D′) are empty for $\tau \gg 1$.

Lemma 4.12 (Type (A3′)) *Let* $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ *be a SFT limit of curves in* $\mathcal{M}^{J^\tau}(c_{\mathbf{p}',\mathbf{q}'}; c_{\mathbf{p},\mathbf{q}})$ *that are not in Type(A1') and (A2'). Then* u_∞ *is empty.*

Proof There is $v \in V(\mathcal{T})$ such that $c_{\mathbf{p}',\mathbf{q}'}$ is a negative asymptote of u_v. By boundary condition, $c_{\mathbf{p},\mathbf{q}}$ cannot be an asymptote of u_v. The existence of u_v violates Lemma 4.5. □

By Lemma 4.5 again, we have.

Lemma 4.13 (Type (D′)) *Let* $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ *be a SFT limit of curves in* $\mathcal{M}^{J^\tau}(c_{\mathbf{p},\mathbf{q}}; x)$. *Then* u_∞ *is empty.*

4.4 Local contribution

In this section, we will determine the algebraic count of some moduli of rigid J^--holomorphic curves in $SM^- = T^*P$, using a *cohomological counting* argument.

Let $q_1, q_2, q_3 \in P$ be three generic points such that $\bigcup_i \Lambda_{q_i}$ satisfies Corollary 3.23. Let $\mathbf{q}_i \in \mathbf{P}$ be a lift of q_i for $i = 1, 2, 3$. Let $\mathbf{J}^-$ be the almost complex structure on $T^*\mathbf{P}$ that is lifted from J^-. Since the contact form $\theta|_{\partial \mathbf{U}}$ equals to the lift of $\alpha = \theta|_{\partial U}$, by Lemma 3.21, there is a unique Reeb chord $x_{i,j}$ from $\Lambda_{\mathbf{q}_i}$ to $\Lambda_{\mathbf{q}_j}$ such that $|x_{i,j}| = 0$ in the canonical relative grading. Let $\mathbf{q}_i \in CF(T^*_{\mathbf{q}_i}\mathbf{P}, \mathbf{P})$ and $\mathbf{c}_{i,j} \in CF(T^*_{\mathbf{q}_i}\mathbf{P}, \tau_{\mathbf{P}}(T^*_{\mathbf{q}_j}\mathbf{P}))$ be the chains represented by the unique geometric intersection in the respective chain complexes.

We are interested in the algebraic counts of the following moduli spaces

(1) $\mathcal{M}^{\mathbf{J}^-}(\mathbf{q}_1; \mathbf{q}_2, x_{1,2})$, $\mathcal{M}^{\mathbf{J}^-}(q_2^\vee; x_{1,2}, \mathbf{q}_1^\vee)$ and $\mathcal{M}^{\mathbf{J}^-}(\varnothing; \mathbf{q}_2, x_{1,2}, \mathbf{q}_1^\vee)$,
(2) $\mathcal{M}^{\mathbf{J}^-}(\mathbf{c}_{3,2}; x_{1,2}, \mathbf{c}_{3,1})$,
(3) $\mathcal{M}^{\mathbf{J}^-}(\mathbf{c}_{1,3}; \mathbf{c}_{2,3}, x_{1,2})$,
(4) $\mathcal{M}^{\mathbf{J}^-}(\varnothing; \mathbf{c}_{2,1}, x_{1,2})$ (Fig. 7).

Theorem 4.14 *The algebraic count of the above moduli spaces are all* ± 1.

Proof of Theorem 4.14 We will apply SFT stretching on the the following “big local model”.

Consider an A_3 Milnor fiber consisting of the plumbing of three copies of T^*S^n. We denote the Lagrangian spheres by S_1, $\mathbf{P}$ and S_3, respectively, where $S_1 \cap S_3 = \varnothing$. We can identify a neighborhood of $\mathbf{P}$ with $\mathbf{U}$. By Hamiltonian isotopy if necessary, we assume that $\mathbf{U} \cap S_j$ is a pair of disjoint cotangent fibers for $j = 1, 3$. We perturb S_1 to S_2 by a perfect Morse function, so that $\mathbf{U} \cap S_2$ is another cotangent fiber.

It will be clear that we should, for $j = 1, 2, 3$, naturally abuse the notation to denote $\mathbf{q}_j \in CF(S_j, \mathbf{P})$, which is the only generator in the corresponding cochain complex. Let $e, pt \in CF(S_1, S_2)$ be the minimum and maximum of the Morse function, respectively, where e represents the identity in cohomology. On the cohomological level, it

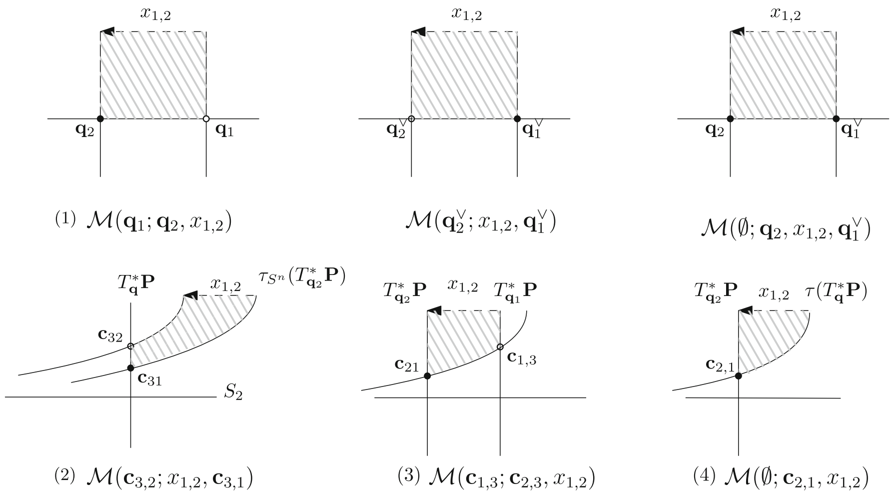

Fig. 7 Six moduli spaces in Theorem 4.14

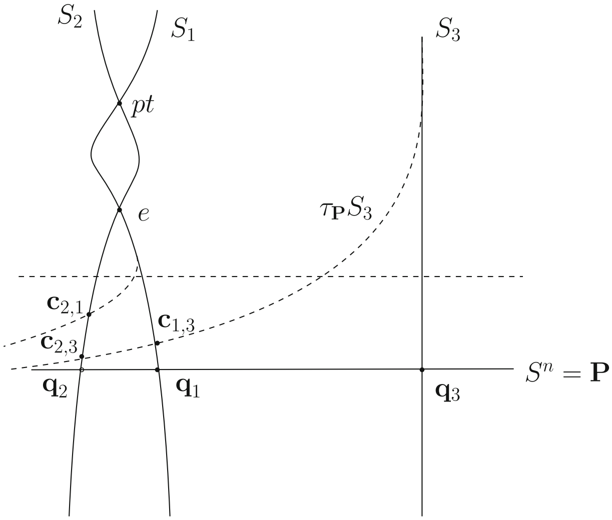

Fig. 8 Big local model before stretch

is clear that $[\mathbf{q}_2][e] = \pm[\mathbf{q}_1]$ and $[e][\mathbf{q}_1^\vee] = \pm[\mathbf{q}_2^\vee]$. This implies the algebraic count

$$\begin{aligned}\#\mathcal{M}(\mathbf{q}_1; \mathbf{q}_2, e) &= \pm 1\\ \#\mathcal{M}(\mathbf{q}_2^\vee; e, \mathbf{q}_1^\vee) &= \pm 1.\end{aligned} \tag{4.19}$$

We now apply the same argument to other cochain complexes. For $i \neq j$, let $\mathbf{c}_{i,j} \in CF^*(S_i, \tau_{\mathbf{P}}(S_j))$ (Fig. 8). be the only generator in their corresponding complex. Again, the multiplication by $[e]$ on $[\mathbf{c}_{1,3}]$ and $[\mathbf{c}_{3,1}]$ yields

$$\#\mathcal{M}(\mathbf{c}_{1,3}; \mathbf{c}_{2,3}, e) = \pm 1, \tag{4.20}$$

$$\#\mathcal{M}(\mathbf{c}_{3,2}; e, \mathbf{c}_{3,1}) = \pm 1. \tag{4.21}$$

For the case of $\mathbf{c}_{2,1} \in CF(S_2, \tau_{\mathbf{P}}(S_1))$, it is immediate from Seidel's exact sequence that rank $HF(S_2, \tau_{\mathbf{P}}(S_1)) = 1$, concentrated on degree 0. $CF(S_2, \tau_S S_1)$ has two additional generators $|\mathbf{c}_{2,1}| = n-1$ and $|pt| = n$, which cancel each other. Therefore, one has

$$\#\mathcal{M}(pt; \mathbf{c}_{2,1}) = \pm 1 \tag{4.22}$$

To deduce Theorem 4.14, we perform a neck-stretching along $\partial \mathbf{U}$. It means that we choose a family of almost complex structure $\mathbf{J}^\tau$ adapted to $\partial \mathbf{U}$ and see how the $\mathbf{J}^\tau$-holomorphic curves converge as τ goes to infinity. We require that the limiting almost complex structure on $S\mathbf{U}$ coincides with $\mathbf{J}^-$ and we denote the limiting almost complex structure outside $\mathbf{U}$ by $\mathbf{J}^+$. S_1 and S_2 give two fibers in $\mathbf{U}$, and every holomorphic curve in $\mathcal{M}^{\mathbf{J}^\tau}(\mathbf{q}_1; \mathbf{q}_2, e)$ will converge, in the $\mathbf{U}$ part, to a curve in $\mathcal{M}^{\mathbf{J}^-}(\mathbf{q}_1; \mathbf{q}_2, x_{1,2})$ (see Lemma 4.9, where the direction of the strip-like ends are switched). This implies

$$(\#\mathcal{M}^{\mathbf{J}^-}(\mathbf{q}_1; \mathbf{q}_2, x_{1,2})) \cdot (\#\mathcal{M}^{\mathbf{J}^+}(x_{1,2}; e)) = \#\mathcal{M}^{\mathbf{J}^\tau}(\mathbf{q}_1; \mathbf{q}_2, e) = \pm 1.$$

Since all counts are integers, it follows that $\#\mathcal{M}^{J^-}(\mathbf{q}_1; \mathbf{q}_2, x_{1,2}) = \pm 1$ which implies the same is true for $\#\mathcal{M}^{J^-}(\mathbf{q}_2^\vee; x_{1,2}, \mathbf{q}_1^\vee)$ and $\#\mathcal{M}^{J^-}(\varnothing; \mathbf{q}_2, x_{1,2}, \mathbf{q}_1^\vee)$.

The same stretching argument, along with (4.20), (4.21) and (4.22) yields

$$(\#\mathcal{M}^{\mathbf{J}^-}(\mathbf{c}_{1,3}; \mathbf{c}_{2,3}, x_{1,2})) \cdot (\#\mathcal{M}^{\mathbf{J}^+}(x_{1,2}; e)) = \#\mathcal{M}^{\mathbf{J}^\tau}(\mathbf{c}_{1,3}; \mathbf{c}_{2,3}, e) = \pm 1, \tag{4.23}$$

$$(\#\mathcal{M}^{\mathbf{J}^-}(\mathbf{c}_{3,2}; x_{1,2}, \mathbf{c}_{31})) \cdot (\#\mathcal{M}^{\mathbf{J}^+}(x_{1,2}; e)) = \#\mathcal{M}^{\mathbf{J}^\tau}(\mathbf{c}_{3,2}; e, \mathbf{c}_{3,1}) = \pm 1, \tag{4.24}$$

$$(\#\mathcal{M}^{\mathbf{J}^-}(\varnothing; \mathbf{c}_{2,1}, x_{1,2})) \cdot (\#\mathcal{M}^{\mathbf{J}^+}(x_{1,2}, pt; \varnothing)) = \#\mathcal{M}^{\mathbf{J}^\tau}(pt; \mathbf{c}_{2,1}) = \pm 1. \tag{4.25}$$

which give the remaining algebraic counts.

Finally, notice that even though S_2 is obtained by a perturbation of S_1, we can actually Hamiltonian isotope S_2 so that $S_2 \cap P$ is the preassigned q_2 and there is no new intersection between S_2 and S_1, S_3 being created during the isotopy. With this choice of S_2 and the stretching argument explained above, Theorem 4.14 follows. □

One may define the analogous moduli spaces similarly on T^*P for cotangent fibers $T^*_{q_i}P$. By equivariance, every rigid J^--holomorphic curve lifts to $|\Gamma|$ many rigid $\mathbf{J}^-$-holomorphic curves and every rigid $\mathbf{J}^-$-holomorphic curve descends to a rigid J^--holomorphic curve.

With this understood, we have.

Corollary 4.15 *The algebraic count of the following moduli spaces are* ± 1.

(1) $\mathcal{M}^{J^-}(p'; p, x_{\mathbf{p}',\mathbf{p}})$, $\mathcal{M}^{J^-}(q'^\vee; x_{\mathbf{q},\mathbf{q}'}, q^\vee)$ and $\mathcal{M}^{J^-}(\varnothing; p, x_{\mathbf{q},\mathbf{p}}, q^\vee)$,
(2) $\mathcal{M}^{J^-}(c_{\mathbf{p},\mathbf{q}'}; x_{\mathbf{q},\mathbf{q}'}, c_{\mathbf{p},\mathbf{q}})$
(3) $\mathcal{M}^{J^-}(c_{\mathbf{p}',\mathbf{q}}; c_{\mathbf{p},\mathbf{q}}, x_{\mathbf{p}',\mathbf{p}})$
(4) $\mathcal{M}^{J^-}(\varnothing; c_{\mathbf{p},\mathbf{q}}, x_{\mathbf{q},\mathbf{p}})$

where $x_{\mathbf{p}',\mathbf{p}}$ is the unique Reeb chord of canonical relative grading 0 from $\Lambda_{p'}$ to Λ_p which can be lifted to a Reeb chord from $\Lambda_{\mathbf{p}'}$ to $\Lambda_{\mathbf{p}}$. The definition of $x_{\mathbf{q},\mathbf{q}'}$ and $x_{\mathbf{q},\mathbf{p}}$ are similar.

4.5 Matching differentials

We now are ready to prove Proposition 4.1. The first lemma relates algebraic counts of differentials of Type (A1) and (A1′).

Lemma 4.16 *For $\tau \gg 1$, the algebraic count of following moduli spaces are equal*

- $\mathcal{M}^{J^\tau}(c_{\mathbf{p}',\mathbf{q}}; c_{\mathbf{p},\mathbf{q}})$, *differentials in hom*$(L_0, \tau_P(L_1))$ *from* $c_{\mathbf{p},\mathbf{q}}$ *to* $c_{\mathbf{p}',\mathbf{q}}$,
- $\mathcal{M}^{J^\tau}(\mathbf{p}'; \mathbf{p})$, *differentials in hom*$(L_0, \mathcal{P})$ *from* $\mathbf{p}$ *to* $\mathbf{p}'$

Proof To prove the lemma, we look at the SFT limit of these moduli when τ goes to infinity. Let u^1_∞ and u^2_∞ be a limiting holomorphic building from curves in $\mathcal{M}^{J^\tau}(c_{\mathbf{p}',\mathbf{q}}; c_{\mathbf{p},\mathbf{q}})$ and $\mathcal{M}^{J^\tau}(\mathbf{p}'; \mathbf{p})$, respectively. Lemmas 4.6 and 4.7, u^i_∞ consist of a J^--holomorphic curve $u^i_{v_1}$ and a J^+-holomorphic curve $u^i_{v_2}$. Moreover, $u^i_{v_2}$ lies in $\mathcal{M}^{J^+}(x_{\mathbf{p},\mathbf{p}'}; \varnothing)$ for both i. On the other hand, $u^1_{v_1}$ lies in $\mathcal{M}^{J^-}(c_{\mathbf{p}',\mathbf{q}}; c_{\mathbf{p},\mathbf{q}}, x_{\mathbf{p}',\mathbf{p}})$ and $u^2_{v_1}$ lies in $\mathcal{M}^{J^-}(\mathbf{p}'; \mathbf{p}, x_{\mathbf{p}',\mathbf{p}})$.

Therefore, for $\tau \gg 1$,

$$
\begin{aligned}
&\#\mathcal{M}^{J^\tau}(\mathbf{p}', \mathbf{p}) \\
&= \#\mathcal{M}^{J^+}(x_{\mathbf{p},\mathbf{p}'}; \varnothing) \cdot \#\mathcal{M}^{J^-}(p'; p, x_{\mathbf{p}',\mathbf{p}}) \\
&= \#\mathcal{M}^{J^+}(x_{\mathbf{p},\mathbf{p}'}; \varnothing) \cdot \#\mathcal{M}^{J^-}(c_{\mathbf{p}',\mathbf{q}}; c_{\mathbf{p},\mathbf{q}}, x_{\mathbf{p}',\mathbf{p}}) \\
&= \#\mathcal{M}^{J^\tau}(c_{\mathbf{p}',\mathbf{q}}; c_{\mathbf{p},\mathbf{q}})
\end{aligned}
$$

where the second equality uses Corollary 4.15 (1) and (3). □

Similarly, we compare the differentials of Type (A2) and (A2′).

Lemma 4.17 *For $\tau \gg 1$, the algebraic count of following moduli spaces are equal*

- $\mathcal{M}^{J^\tau}(c_{\mathbf{p},\mathbf{q}'}; c_{\mathbf{p},\mathbf{q}})$,
- $\mathcal{M}^{J^\tau}(\mathbf{q}'^\vee; \mathbf{q}^\vee)$.

Proof The proof is almost word-by-word taken from Lemma 4.16. Lemma 4.6, 4.7 and Corollary 4.15 (1) and (2) implies

$$
\begin{aligned}
&\#\mathcal{M}^{J^\tau}(\mathbf{q}'^\vee; \mathbf{q}^\vee) \\
&= \#\mathcal{M}^{J^+}(x_{\mathbf{q},\mathbf{q}'}) \cdot \#\mathcal{M}^{J^-}(q'^\vee; x_{\mathbf{q},\mathbf{q}'}, q^\vee) \\
&= \#\mathcal{M}^{J^+}(x_{\mathbf{q},\mathbf{q}'}) \cdot \#\mathcal{M}^{J^-}(c_{\mathbf{p},\mathbf{q}'}; x_{\mathbf{q},\mathbf{q}'}, c_{\mathbf{p},\mathbf{q}}) \\
&= \#\mathcal{M}^{J^\tau}(c_{\mathbf{p},\mathbf{q}'}; c_{\mathbf{p},\mathbf{q}})
\end{aligned}
$$

□

The last lemma addresses differentials of Type (C) and (C′).

Lemma 4.18 *For $\tau \gg 1$, the algebraic count of following moduli spaces are equal*

- $\mathcal{M}^{J^\tau}(x; \mathbf{q}^\vee, \mathbf{p})$, *for some $x \in CF^*(L_0, L_1)$ represented by an intersection outside U,*
- $\mathcal{M}^{J^\tau}(x; c_{\mathbf{p},\mathbf{q}})$.

Proof The strategy is still similar. Apply the same neck-stretching as in Lemmas 4.16 and 4.17, one obtains a building consisting of a triangle and a bigon for $\mathcal{M}^{J^\tau}(x; \mathbf{q}^\vee, \mathbf{p})$, thanks to Lemma 4.9; and a building consisting of two bigons for $\mathcal{M}^{J^\tau}(x; c_{\mathbf{p},\mathbf{q}})$ from Lemma 4.11. Therefore

$$
\begin{aligned}
&\#\mathcal{M}^{J^\tau}(x; \mathbf{q}^\vee, \mathbf{p}) \\
&= \#\mathcal{M}^{J^+}(x, x_{\mathbf{q},\mathbf{p}}; \varnothing) \cdot \#\mathcal{M}^{J^-}(\varnothing; p, x_{\mathbf{q},\mathbf{p}}, q^\vee) \\
&= \#\mathcal{M}^{J^+}(x, x_{\mathbf{q},\mathbf{p}}; \varnothing) \cdot \#\mathcal{M}^{J^-}(\varnothing; c_{\mathbf{p},\mathbf{q}}, x_{\mathbf{q},\mathbf{p}}) \\
&= \#\mathcal{M}^{J^\tau}(x; c_{\mathbf{p},\mathbf{q}})
\end{aligned}
$$

where the second equality uses Corollary 4.15 (1) and (4). □

As the end product of this section, we have.

Proof of Proposition 4.1 For $\tau \gg 1$, the differential on C_0 and C_1 can be identified by Lemmas 4.16, 4.17, 4.8, 4.18 and 4.5. □

The proof of Proposition 4.1 when $\mathrm{char}(\mathbb{K}) \neq 2$ is given in "Appendix A".

5 Categorical level identification

In this section, we want to prove Theorem 1.2 by showing the following:

Theorem 5.1 *For any object $\mathcal{E}^1 \in Ob(\mathcal{F})$, we can perform a Hamiltonian perturbation for $\mathcal{E}^1$ to obtain another object $(\mathcal{E}^1)'$ of $\mathcal{F}$ such that there is a degree zero cochain $c_{\mathcal{D}} \in hom^0_{\mathcal{F}^{\mathrm{perf}}}(\tau_P((\mathcal{E}^1)'), T_{\mathcal{P}}(\mathcal{E}^1))$ so that $c_{\mathcal{D}}$ is a cocycle, and*

$$\mu^2(c_{\mathcal{D}}, \cdot) : hom^0_{\mathcal{F}^{\mathrm{perf}}}(\mathcal{E}^0, \tau_P((\mathcal{E}^1)')) \to hom^0_{\mathcal{F}^{\mathrm{perf}}}(\mathcal{E}^0, T_{\mathcal{P}}(\mathcal{E}^1)) \tag{5.1}$$

is a quasi-isomorphism for all $\mathcal{E}^0 \in Ob(\mathcal{F})$

In particular, $\tau_P(\mathcal{E}^1) \simeq \tau_P((\mathcal{E}^1)') \simeq T_{\mathcal{P}}(\mathcal{E}^1)$ as perfect A_∞ right $\mathcal{F}$-modules.

The overall strategy goes as follows. By Proposition 4.1, Corollary 2.18 and the fact that Hamiltonian isotopic objects are quasi-isomorphic, we know that

$$H(hom_{\mathcal{F}^{\mathrm{perf}}}(\tau_P((\mathcal{E}^1)'), T_{\mathcal{P}}(\mathcal{E}^1))) = HF(\tau_P(\mathcal{E}^1), \tau_P(\mathcal{E}^1)) \tag{5.2}$$

Our goal is to pick an appropriate non-exact degree zero cocycle $c_{\mathcal{D}} \in hom^0_{\mathcal{F}^{\mathrm{perf}}}(\tau_P((\mathcal{E}^1)'), T_{\mathcal{P}}(\mathcal{E}^1))$, and check that $\mu^2(c_{\mathcal{D}}, -)$ is a quasi-isomorphism for all $\mathcal{E}^0 \in Ob(\mathcal{F})$ [see (5.1)]. By a Hamiltonian perturbation if necessary, it suffices to check the equality for those $\mathcal{E}^0$ such that L_0 intersects L_1, $(L_1)'$ and P transversally. This allows us to apply neck-stretching along ∂U to compute $\mu^2(c_{\mathcal{D}}, \cdot)$ for $\tau \gg 1$ (see Sect. 3.8).

The discussion in this section works for fields $\mathbb{K}$ of arbitrary characteristics, even though we didn't pay exclusive attention to signs.

Again, let us give a sketch of this section in hope of rescuing discouraged readers from the daunting details and notations. As pointed out in the introduction, we will pursue the generator that comes from L and the Dehn twist of a perturbation of L, which represents the fundamental class of $CF(L, L)$ before the Dehn twist. This is not a cocycle in $\mathcal{D}$, and we computed its differential in 5.1.1. To offset them, we use the tensor product component in $\mathcal{D}$, whose differential, as a product in the Fukaya category, is computed in 5.1.2, which eventually yields the desired cocycle $c_{\mathcal{D}}$. After studying more of the A_∞-structure, we verify $c_{\mathcal{D}}$ gives the desired quasi-isomorphism (1.1).

The reader should note that we postpone all issues of orientations to the "Appendix", but as it turns out, the content in this section depends on analysis of signs minimally.

5.1 Hunting for degree zero cocycles

To find a degree zero cocycle, we need to first analyze the differential of $hom_{\mathcal{F}^{\text{perf}}}(\tau_P((\mathcal{E}^1)'), T_{\mathcal{P}}(\mathcal{E}^1))$ by neck-stretching. The discussion in this section works for field $\mathbb{K}$ of **arbitrary** characteristics.

Let L_1' be a C^2-small Hamiltonian push-off of L_1 such that $L_1' \cap U$ is a union of cotangent fibers. Let $q_1, \ldots, q_{d_{L_1}} \in CF(L_1, P)$ and $q_1', \ldots, q_{d_{L_1}}' \in CF(L_1', P)$ be the cochain representatives of the geometric intersection points, where $d_{L_1} = \#(P \cap L_1) = \#(P \cap L_1')$. We also number the intersection points so that $d_P(q_i, q_i') \ll \epsilon$ in the standard quotient round metric. Let $\Lambda_{q_i}, \Lambda_{q_j'} \subset \partial U$ be the cospheres at q_i and q_j', respectively. We assume q_i, q_j' satisfy Corollary 3.23. Fix $\mathbf{q}_i, \mathbf{q}_j'$ be a lift of q_i, q_j', respectively, for all i, j. Our focus will be the cochain complex

$$\begin{aligned}\mathcal{D} := hom_{\mathcal{F}^{\text{perf}}}(\tau_P((\mathcal{E}^1)'), T_{\mathcal{P}}(\mathcal{E}^1)) = (CF(\mathcal{P}, \mathcal{E}^1) \\ \otimes_\Gamma CF(\tau_P((\mathcal{E}^1)'), \mathcal{P}))[1] \oplus CF(\tau((\mathcal{E}^1)'), \mathcal{E}^1)\end{aligned} \tag{5.3}$$

which is generated by elements supported at the intersection points

$$\begin{cases} q_i^\vee \otimes \tau_P(q_j'), & \text{for } i, j = 1, \ldots, d_{L_1} \\ c_{i,g,j}^\vee := c_{\mathbf{q}_i, g\mathbf{q}_j'}^\vee, & \text{for } g \in \Gamma, i, j = 1, \ldots, d_{L_1} \\ w_k, & \text{for } k = 1, \ldots, \#(L_1' \cap L_1) \end{cases} \tag{5.4}$$

The first two kinds of intersection points are inside U while $\{w_k\}$ are outside U. Elements supported at $c_{i,g,j}^\vee$ and w_k are given by

$$Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{c_{\mathbf{q}_i, g\mathbf{q}_j'}}, \mathcal{E}^1_{c_{\mathbf{q}_i, g\mathbf{q}_j'}}) \text{ and } Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{w_k}, \mathcal{E}^1_{w_k}) \tag{5.5}$$

respectively. On the other hand, the elements supported at $q_i^\vee \otimes \tau_P(q_j')$ are generated by

$$(\psi^2 \otimes \mathbf{q}_i^\vee) \otimes (g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1), \text{ for } \psi^2 \in \mathcal{E}^1_{q_i}, \psi^1 \in Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{\tau_P(q_j')}, \mathbb{K}), g \in \Gamma \quad (5.6)$$

Here we use the commutativity $\pi(\tau_{\mathbf{P}}(\mathbf{q}_j')) = \tau_P(\pi(\mathbf{q}_j')) = \tau_P(q_j')$.

Lemma 5.2 *With respect to canonical relative grading, we have*

$$\begin{cases} |q_i^\vee| = 0, & \textit{for } q_i^\vee \in hom(P, T^*_{q_i}P) \\ |\tau_P(q_j')| = 1, & \textit{for } \tau_P(q_j') \in hom(\tau_P(T^*_{q_j'}P), P) \\ |c^\vee_{i,g,j}| = 1, & \textit{for } c^\vee_{i,g,j} = \pi(\tau_{\mathbf{P}}(T^*_{g\mathbf{q}_j'}\mathbf{P}) \cap T^*_{\mathbf{q}_i}\mathbf{P}) \in hom(\tau_P(T^*_{q_j'}P), T^*_{q_i}P) \end{cases} \quad (5.7)$$

Proof The fact that $|q_i^\vee| = 0$ follows from the definition of canonical relative grading (3.8). $|c^\vee_{i,g,j}| = 1$ follows from $|c_{i,g,j}| = n-1$ (see Lemma 4.5). Finally, from the long exact sequence

$$HF^k(P, T^*_{q_j'}P) \to HF^k(P, \tau_P(T^*_{q_j'}P)) \to HF^{k+1}(P, P) \to HF^{k+1}(P, T^*_{q_j'}P) \quad (5.8)$$

and the fact that $HF(P, \tau_P(T^*_{q_j'}P))$ has rank 1, we know that $HF^0(P, P) \simeq HF^0(P, T^*_{q_j'}P)$, and $HF^k(P, \tau_P(T^*_{q_j'}P)) \to HF^{k+1}(P, P)$ is an isomorphism when $k = n-1$. Therefore, $|\tau_P(q_j')^\vee| = n-1$ and $|\tau_P(q_j')| = n - |\tau_P(q_j')^\vee| = 1$. □

Without loss of generality, we assume that there is a unique w_k with degree 0 and we denote it by e_L. All other w_k has $|w_k| > 0$. With generators understood, we now recall that the differential for element ψ_x supported at $x = c^\vee_{i,g,j}$ or $x = w_k$ is given by $\mu^1(\psi_x) = \mu^1_{\mathcal{F}}(\psi_x)$, and for element supported at $q_i^\vee \otimes \tau_P(q_j')$ is given by [see (2.87)]

$$\begin{aligned} \mu^1_{\mathcal{D}}(\psi^2 \otimes \mathbf{q}_i^\vee \otimes g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1) = &(-1)^{|g\tau_{\mathbf{P}}(\mathbf{q}_j')|}\mu^1_{\mathcal{F}}(\psi^2 \otimes \mathbf{q}_i^\vee) \otimes (g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1) \\ &+ (\psi^2 \otimes \mathbf{q}_i^\vee) \otimes \mu^1_{\mathcal{F}}(g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1) \\ &+ \mu^2_{\mathcal{F}}(\psi^2 \otimes \mathbf{q}_i^\vee, g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1) \end{aligned} \quad (5.9)$$

Our focus will be put on $\mu^1_{\mathcal{F}}(\psi_{e_L})$ and $\mu^2_{\mathcal{F}}(\psi^2 \otimes \mathbf{q}_i^\vee, g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1)$.

5.1.1 Computing $\mu^1_{\mathcal{F}}(\psi_{e_L})$

Let $h : L_1 \to \mathbb{R}$ be a smooth function such that $dh = \theta|_{L_1}$. We define $h_i := h|_{\Lambda_{q_i}}$ which are constants because L_1 is cylindrical near Λ_{q_i}. Hamiltonian push-off induces $h' : L_1' \to \mathbb{R}$ such that $dh' = \theta|_{L_1'}$ and $h_i' := h'|_{\Lambda_{q_i'}}$ are constants. By possibly reordering the index set of i, we assume that $h_1 \leqslant h_2 \leqslant \cdots \leqslant h_d$. For each i, by relabelling if necessary, we also assume that q_i' is the closest to q_i among points in $\{q_j'\}_{j=1}^{d_{L_1}}$, and $\mathbf{q}_i'$ is the closest to $\mathbf{q}_i$ among points in $\{g\mathbf{q}_i'\}_{g\in\Gamma}$.

We recall from (3.35) that the action of a Reeb chord x from $\Lambda_{q_j'}$ to Λ_{q_i} is given by

$$A(x) := L(x) + h_j' - h_i \tag{5.10}$$

Lemma 5.3 *There is a constant $\epsilon > 0$ depending only on $\{q_i\}_{i=1}^{d_{L_1}}$ and L_1 such that when L_1' is a sufficiently small Hamiltonian push-off of L_1,*

- *$A(x) > \epsilon$ if x is a Reed chord from $\Lambda_{q_j'}$ to Λ_{q_i} and $j > i$, and*
- *$A(x) > \epsilon$ if x is a Reed chord from $\Lambda_{q_i'}$ to Λ_{q_i} but not the shortest one.*

Proof There is a constant $\epsilon > 0$ depending only on $\{q_i\}_{i=1}^{d_{L_1}}$ and L_1 such that $L(x) > 3\epsilon$ if x is either a Reeb chord from Λ_{q_j} to Λ_{q_i} and $i \neq j$, or it is a **non-constant** Reeb chord from Λ_{q_i} to itself. We can choose a small Hamiltonian perturbation such that $L(x) > 2\epsilon$ if either x is a Reeb chord from $\Lambda_{q_j'}$ to Λ_{q_i}, or a non-shortest Reeb chord from $\Lambda_{q_i'}$ to Λ_{q_i}. If $j \geqslant i$, we have $h_j \geqslant h_i$ so we can assume the Hamiltonian chosen is small enough such that $h_j' - h_i > -\epsilon$ and therefore $A(x) = L(x) + h_j' - h_i > \epsilon$ in both cases listed in the lemma. □

For each i, we denote the shortest Reeb chord from $\Lambda_{q_i'}$ to Λ_{q_i} by $x_{i',i}$. In regards to the canonical relative grading, we have $|x_{i',i}| = 0$. Since $\mathbf{q}_i'$ is the closest to $\mathbf{q}_i$ among points in $\{g\mathbf{q}_i'\}_{g\in\Gamma}$, if we lift the Reeb chord $x_{i',i}$ to a Reeb chord starting from $\Lambda_{\mathbf{q}_i'}$, then it ends on $\Lambda_{\mathbf{q}_i}$.

The following Lemmas (5.4, 5.5 and 5.6) concern some moduli of rigid bigon with input being e_L. We start with the case when the output lies outside U.

Lemma 5.4 *For $\tau \gg 1$, rigid elements in $\mathcal{M}^{J^\tau}(w_k; e_L)$ with respect to boundary conditions $(\tau_P(L_1'), L_1)$ and (L_1', L_1) (i.e. they contribute to the differential in $CF(\tau_P(L_1'), L_1)$ and $CF(L_1', L_1)$), respectively, can be canonically identified.*

Proof By the same reasoning as in Lemma 4.8, as τ goes to infinity, the holomorphic building $u_\infty = (u_v)_{v\in V(\mathcal{T})}$ consists of exactly one vertex v and u_v maps to SM^+. The result follows. □

In Lemma 5.3, the ϵ is independent of perturbation. Therefore, we can choose a perturbation such that the action of e_L in $hom(L_1', L_1)$ (and hence in $hom(\tau(L_1'), L_1)$) is less than ϵ. In this case, we have

Lemma 5.5 *Let ϵ satisfy Lemma 5.3. If $A(e_L) < \epsilon$, then for all $j > i$ and $g \in \Gamma$ (or $j = i$ and $g \neq 1_\Gamma$), there is no rigid element in $\mathcal{M}^{J^\tau}(c_{i,g,j}^\vee; e_L)$ for $\tau \gg 1$.*

Proof Suppose not, then we will have a holomorphic building $u_\infty = (u_v)_{v\in V(\mathcal{T})}$ as τ goes to infinity. Let u_{v_1} be the J^--holomorphic curve such that $c_{i,g,j}^\vee$ is an asymptote of u_{v_1}. One can argue as in Lemma 4.11 to show that u_{v_1} has exactly one positive Reeb chord asymptote x. Moreover, x can be lifted to a Reeb chord from $\Lambda_{g\mathbf{q}_j'}$ to $\Lambda_{\mathbf{q}_i}$

by boundary condition. When $j > i$ and $g \in \Gamma$ (or $j = i$ and $g \neq 1_\Gamma$), we have $A(x) > \epsilon$ by Lemma 5.3. Since $A(e_L) < \epsilon$ by assumption, we get a contradiction by Lemma 3.15. □

Lemma 5.6 *For L_1' sufficently close to L_1 and $\tau \gg 1$, the algebraic count of rigid elements in $\mathcal{M}^{J^\tau}(c^\vee_{i,1_\Gamma,i}; e_L)$ is ± 1.*

Proof Similar to previous discussions, every limiting holomorphic building $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ from strips in $\mathcal{M}^{J^\tau}(c^\vee_{i,1_\Gamma,i}; e_L)$ consists of two vertices (see Lemma 4.11). By boundary condition, the bottom level curve u_{v_1} lies in $\mathcal{M}^{J^-}(c^\vee_{i,1_\Gamma,i}; x_{i',i})$, which has algebraic count ± 1 by Corollary 4.15(4). Therefore, it suffices to determine the algebraic count of $\mathcal{M}^{J^+}(x_{i',i}; e_L)$.

We consider the rigid elements in the moduli $\mathcal{M}(q_i^\vee; e_L, (q_i')^\vee)$ for a compatible almost complex structure J, which is responsible to the $q_i^\vee$-coefficient of $\mu^2(e_L, (q_i')^\vee)$ for the operation $\mu^2(\cdot, \cdot) : hom(L_1', L_1) \times hom(P, L_1') \to hom(P, L_1)$. Therefore, it has algebraic count ± 1 with respect to J when L_1' is C^2-close to L_1.

Next, we will use a cascade (homotopy) type argument which goes back to Floer and argue that the algebraic count of $\mathcal{M}^{J^\tau}(q_i^\vee; e_L, (q_i')^\vee)$ is ± 1 for all $\tau < \infty$. The difficulty lies in that neither $q_i^\vee$ or $(q_i')^\vee$ is a cocycle, so the cohomological arguments would not work here. A detailed account for a cascade (homotopy) type argument involving higher multiplications can be found in, for example, [45] (see also [2,46, Section 10*e*]).

Let us recall the overall strategy of the cascade argument tailored for our situation. Pick a path of compatible almost complex structures $(J_t)_{t \in [0,\infty)}$ from J to J^τ for some finite time τ. For a generic path of almost complex structure $(J_t)_{t \in [0,\infty)}$, there are finitely many $0 < t_1 < \cdots < t_k < 1$ such that there exists J_{t_l} stable maps with input $e_L, (q_i')^\vee$, output $q_i^\vee$ and consisting of two components. In our case, they consist of a J_{t_l}-holomorphic triangle and a bigon, respectively. Moreover, one of the components must be of virtual dimension 0, and the other one is of dimension -1. In this case, we say *a bifurcation occurs at* t_l, and denote the component of virtual dimension -1 as u.

If a bifurcation occurs at t_l, then $\mathcal{M}^{J_t}(q_i^\vee; e_L, (q_i')^\vee)$ has the same diffeomorphism type when $t \in (t - \epsilon, t_l)$ for some small $\epsilon > 0$. The *birth-death* bifurcation cancels a pair of $J_{t_l - \epsilon}$-triangles at time t_l; and the *death-birth* bifurcation creates a pair of $J_{t_l+\epsilon}$-triangles at the time t_l. In either case, there is pair of stable J_{t_l}-stable triangles. When t approaches t_l from the right, we get the corresponding cobordisms. The change of algebraic count from $\mathcal{M}^{J_{t_l-\epsilon}}(q_i^\vee; e_L, (q_i')^\vee)$ to $\mathcal{M}^{J_{t_l+\epsilon}}(q_i^\vee; e_L, (q_i')^\vee)$ is called the *contribution to $\mathcal{M}^{J_t}(q_i^\vee; e_L, (q_i')^\vee)$ by the bifuration at time t_l.*

Therefore, to show that the algebraic count persists to be ± 1 crossing t_l, we will analyze each bifurcation moment t_l below and prove the contribution to $\mathcal{M}^{J_t}(q_i^\vee; e_L, (q_i')^\vee)$ is zero. For simplicity we let $l = 1$. Since there are exactly two irreducible components at $t = t_1$, one of them has to has virtual dimension 0 and the other one has dimension -1. Let u denote the component of virtual dimension -1 (it can be either a strip or a triangle), and we divide the possible stable J_{t_1}-holomorphic triangles into three cases:

(i) both $q_i^\vee$ and $(q_i')^\vee$ are asymptotes of u;

(ii) exactly one of $q_i^\vee$ and $(q_i')^\vee$ is an asymptote of u;
(iii) neither of $q_i^\vee$ nor $(q_i')^\vee$ is an asymptote of u.

Case (i) If both $q_i^\vee$ and $(q_i')^\vee$ are asymptotes of u, then the last asymptote x of u must be a generator of $CF(L_1', L_1)$ by boundary condition. Moreover, x is a degree 1 element of $CF(L_1', L_1)$ because $\text{virdim}(u) = -1$ and $|e_L| = 0$. This bifurcation contributes to a change in the algebraic count of $\mathcal{M}^{J_t}(q_i^\vee; e_L, (q_i')^\vee)$ by the algebraic count of rigid elements from $\mathcal{M}^{J_{t_1}}(x; e_L)$ (when $t > t_1$, the moduli $\mathcal{M}^{J_{t_1}}(x; e_L)$ and $\mathcal{M}^{J_{t_1}}(q_i^\vee; x, (q_i')^\vee)$ glue together to give a change). However, the algebraic count of rigid elements from $\mathcal{M}^{J_{t_1}}(x; e_L)$ is zero because e_L is a cocycle.
Case (ii) If exactly one of $q_i^\vee$ and $(q_i')^\vee$ is an asymptote of u, then P is a Lagrangian boundary condition of one of the component of $\partial\Sigma_u$, where Σ_u is the domain of u. By this boundary component, there is another point q_j or q_j' for some $j \neq i$ which is an asymptote of u. Since there is a lower bound between the distance from q_i (or q_i') to q_j (or q_j') for $j \neq i$, we can apply monotonicity Lemma at an appropriate point in $Im(u) \cap P$ to get a constant $\delta > 0$ depending only on $\{q_i\}_{i=1}^{d_{L_1}}$ but not L_1' such that the energy $E_\omega(u) > \delta$. If we chose L_1' to be sufficiently close to L_1 such that $A(e_L) + A((q_i')^\vee) - A(q_i^\vee) < \delta$, then for u to contribute to a change of algebraic count of $\mathcal{M}^{J_t}(q_i^\vee; e_L, (q_i')^\vee)$, u has to be glued with a rigid J_{t_1}-holomorphic curve of negative energy, which does not exist.
Case (iii) If none of $q_i^\vee$ and $(q_i')^\vee$ are asymptotes of u, then u is a bigon with one asymptote being e_L and the other asymptote, denoted by x, being a generator of $CF(L_1', L_1)$. Moreover, $|x| = 0$ because $\text{virdim}(u) = -1$. It is a contradiction because e_L is the only generator of $CF(L_1', L_1)$ with degree 0 and constant maps have virtual dimension 0.

As a result, no bifurcation can possibly contribute to a change to the algebraic count and $\#\mathcal{M}^{J^\tau}(q_i^\vee; e_L, (q_i')^\vee) = \pm 1$ for all τ. By letting τ go to infinity, the argument in Lemma 4.9 implies that the limiting holomorphic building $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ consist of two vertices. Moreover, we have $u_{v_1} \in \mathcal{M}^{J^-}(q_i^\vee; x_{i',i}, (q_i')^\vee)$ and $u_{v_2} \in \mathcal{M}^{J^+}(x_{i',i}; e_L)$. It implies that the algebraic count of rigid element in $\mathcal{M}^{J^+}(x_{i',i}; e_L)$ is ± 1. The proof finishes. □

Remark 5.7 The fact that the algebraic count of $\mathcal{M}^{J^+}(x_{i',i}; e_L)$ is ± 1 will be used in Proposition 5.8 again.

Let us take local systems on the Lagrangians into account. Let $\mathcal{E}'$, $(\mathcal{E}^1)'$ be local systems supported on L_1, L_1', respectively. Using the Hamiltonian push-off, we have the identifications

$$\tau_P((\mathcal{E}^1)')_{w_k} \simeq (\mathcal{E}^1)'_{w_k} \simeq \mathcal{E}^1_{w_k}, \text{ and } \tau_P((\mathcal{E}^1)')_{c_{\mathbf{q}_i, g\mathbf{q}_j'}} \simeq \mathcal{E}^1_{c_{\mathbf{q}_i, g\mathbf{q}_j'}} \tag{5.11}$$

for all w_k and $c_{\mathbf{q}_i, g\mathbf{q}_j}$. In particular, we can define $t_{\mathcal{D}}$ to be the identity morphism supported at the intersection underlying e_L, but as a morphism, it is written as:

$$t_{\mathcal{D}} := id \in Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{e_L}, \mathcal{E}^1_{e_L}) \subset hom_{\mathcal{F}^{\text{perf}}}(\tau_P((\mathcal{E}^1)'), T_{\mathcal{P}}(\mathcal{E}^1)) = \mathcal{D} \tag{5.12}$$

We also denote $e_{\mathcal{E}}$ as

$$e_{\mathcal{E}} := id \in Hom\left((\mathcal{E}^1)', \mathcal{E}'\right). \tag{5.13}$$

Geometrically, both $t_{\mathfrak{D}}$ and $e_{\mathcal{E}}$ are supported at the same intersection point and represents the same identity morphism between the stalks. $t_{\mathfrak{D}}$ can be regarded as a chain-level preimage of the $\mathcal{E}$ under the (Poincaré) dualized Seidel's exact sequence, hence has no guarantee to be closed.

Let us take local systems on the Lagrangians into account. Since $\pi_1(U \cap L_1) = 1$, we can identify stalks of the local system $\mathcal{E}^1_p$ over each $p \in U \cap L_1$ using the flat connection (equivalently, assume the connection is trivial in $U \cap L_1$). Similary, identify all $(\mathcal{E}^1)'_{p'}$ for $p' \in U \cap L'_1$. This also induces an identification of stalks on $\tau_P(T^*_q P)$, since local systems therein are pushforwards of the ones over a fiber.

We can now summarize the previous lemmss.

Proposition 5.8 *For L'_1 sufficiently close to L_1 and $\tau \gg 1$, we have*

$$\mu^1(t_{\mathfrak{D}}) = \sum_{i,j,g} \psi_{c^\vee_{i,g,j}} \tag{5.14}$$

where $\psi_{c^\vee_{i,g,j}} \in Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{c_{\mathbf{q}_i, g\mathbf{q}'_j}}, \mathcal{E}^1_{c_{\mathbf{q}_i, g\mathbf{q}'_j}})$ and

$$\begin{cases} \psi_{c^\vee_{i,g,j}} = 0 \text{ if } j > i \text{ and } g \in \Gamma (\text{or } j = i \text{ and } g \neq 1_\Gamma) \\ \psi_{c^\vee_{i,1_\Gamma,i}} = \pm id \in Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{c_{\mathbf{q}_i, \mathbf{q}'_i}}, \mathcal{E}^1_{c_{\mathbf{q}_i, \mathbf{q}'_i}}) \end{cases} \tag{5.15}$$

Proof By Lemma 5.4 and the fact that $e_{\mathcal{E}}$ is a cocycle in $CF((\mathcal{E}^1)', \mathcal{E}^1)$, we know that $\mu^1(t_{\mathfrak{D}}) = \sum_{i,j,g} \psi_{c^\vee_{i,g,j}}$. The fact that $\psi_{c^\vee_{i,g,j}} = 0$ if $j > i$ and $g \in \Gamma$ (or $j = i$ and $g \neq 1_\Gamma$) follows from Lemma 5.5. Finally, to see that $\psi_{c^\vee_{i,1_\Gamma,i}} = id$ we need to understand the moduli $\mathcal{M}^{J^\tau}(c^\vee_{i,1_\Gamma,i}; e_L)$ and the parallel transport maps given by the rigid elements in it.

Consider the holomorphic building when $\tau = \infty$, we have two components $u_1 \in \mathcal{M}^{J^-}(q_i^\vee; x_{i',i}, (q'_i)^\vee)$ and $u_2 \in \mathcal{M}^{J^+}(x_{i',i}; e_L)$ by Lemma 4.9 and Remark 4.10. When L'_1 is sufficiently C^2-close to L_1, the action of u_1, u_2 can be as small as we want. It implies that, by monotonicity lemma, u_2 lies in a Weinstein neighborhood of L_1.

It in turn implies that, for each strip u_2 in the limit, the associated output is $\psi_{i',i} = \pm id$ when the input at the point e_L is $t_{\mathfrak{D}}$ (the sign of $\psi_{i',i}$ supported on $x_{i',i}$ depends on the sign of u_2). This is because we have identified the stalks of $\mathcal{E}^1$ and $(\mathcal{E}^1)'$ at the point e_L, and the associated parallel transports $I_{\partial_0 u}$ and $I_{\partial_1 u}$ on their respective boundary conditions are inverse to each other (in fact, the strip itself provides an isotopy after projecting to L_1 in the Weinstein neighborhood). Since we have proved that the algebraic count of $\mathcal{M}^{J^+}(x_{i',i}; e_L)$ is ± 1 (see Remark 5.7), the associated output by all elements in $\mathcal{M}^{J^+}(x_{i',i}; e_L)$ is $\pm id$, when the input at e_L is $t_{\mathfrak{D}}$.

To get the proposition, we now replace u_1 by $u'_1 \in \mathcal{M}^{J^\tau}(c^\vee_{i,1_\Gamma,i}; x_{i',i})$. As explained earlier, we have identified the fibers of the local systems of $\mathcal{E}^1$ and $\tau_P(\mathcal{E}^1)'$ at $c_{\mathbf{q}_i, \mathbf{q}'_i}$.

Since the parallel transports of $\mathcal{E}^1$ and $\tau_P(\mathcal{E}^1)'$ inside U are trivial, if the input at $x_{i',i}$ is $\pm id$, so is the output. By Lemma 5.6, the algebraic count of $\mathcal{M}^{J^\tau}(c^\vee_{i,1_\Gamma,i}; x_{i',i})$ is ± 1 and each strip contributes $\pm id$ (and the sign of $\pm id$ only depends on the sign of the strip), therefore, the total countribution is $\pm id$, as desired. □

Remark 5.9 In summary, when L_1' is sufficiently close to L_1, e_L being a cohomological unit is responsible for the algebraic count of $\mathcal{M}^{J^\tau}(q_i^\vee; e_L, (q_i')^\vee)$ being ± 1 and hence the $q_i^\vee$-coefficient of $\mu^2(e_L, (q_i')^\vee)$ being ± 1. On the other hand, $e_{\mathcal{E}}$ being a cohomological unit is responsible for the $\mathbf{q}_i^\vee$-coefficient of $\mu^2(e_{\mathcal{E}}, (\mathbf{q}_i')^\vee)$ being 1. Lemma 5.6 and Proposition 5.8 are obtained by replacing the bottom level curves at the SFT limit.

5.1.2 Computing $\mu^2_{\mathcal{F}}(\psi^2 \otimes \mathrm{q}_i^\vee, g\tau_{\mathrm{P}}(\mathrm{q}_j') \otimes \psi^1)$

Next, we want to study $\mu^1_{\mathcal{D}}((\psi^2 \otimes \mathbf{q}_i^\vee) \otimes (g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1))$ [see (5.6), (5.9)]. In particular, we want to focus on the term $\mu^2_{\mathcal{F}}(\psi^2 \otimes \mathbf{q}_i^\vee, g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1)$ so we need to discuss the moduli $\mathcal{M}(c^\vee_{i,g,j}; q_i^\vee, \tau_P(q_j'))$ and $\mathcal{M}(w_k; q_i^\vee, \tau_P(q_j'))$.

Lemma 5.10 *For $\tau \gg 1$, there is no rigid element in $\mathcal{M}^{J^\tau}(w_k; q_i^\vee, \tau_P(q_j'))$.*

Proof We argue by contradiction as before. Let $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ be a limiting holomorphic building. By boundary condition, there is $v_1 \in V(\mathcal{T})$ such that $q_i^\vee, \tau_P(q_j')$ are asymptotes of u_{v_1}. The other asymptotes of u_{v_1} are positive Reeb chords $y_1, \ldots, y_k$. The virtual dimension of u_{v_1} can be computed using canonical relative grading (Lemma 5.2), and is given by

$$\begin{aligned} \mathrm{virdim}(u_{v_1}) =& n(1-0) - |q_i^\vee| - |\tau_P(q_j')| \\ & - \sum_{s=1}^{k} |y_s| - (1-k) \geqslant n - 0 - 1 - (1-k) > 0 \end{aligned}$$

because $n \geqslant 3$. It contradicts to $\mathrm{virdim}(u_{v_1}) = 0$. □

Lemma 5.11 *For $\tau \gg 1$, there is no rigid element in $\mathcal{M}^{J^\tau}(c^\vee_{\bar{i},\bar{g},\bar{j}}; q_i^\vee, \tau_P(q_j'))$ unless $c_{\bar{i},\bar{g},\bar{j}} = c_{i,g,j}$ for some $g \in \Gamma$.*

Proof Assume $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ be a limiting holomorphic building. If $c_{\bar{i},\bar{g},\bar{j}} \neq c_{i,g,j}$ for all $g \in \Gamma$, then $c_{\bar{i},\bar{g},\bar{j}} \notin T^*_{q_i}P \cap \tau_P(T^*_{q_j'}P)$. By boundary condition, there is $v_1 \in V(\mathcal{T})$ such that $q_i^\vee, \tau_P(q_j')$ are asymptotes of u_{v_1} but $c^\vee_{\bar{i},\bar{g},\bar{j}}$ is **not** an asymptote of u_{v_1}. Therefore, all other asymptotes of u_{v_1} are positive Reeb chords and we get a contradiction as in Lemma 5.10. □

The following lemma computes the μ^2 map with trivial local systems on L_1 and L_1'.

Lemma 5.12 *For $\tau \gg 1$, the $c^\vee_{i,h,j}$-coefficient of $\mu^2(\mathbf{q}_i^\vee, g\tau_P(\mathbf{q}_j'))$ is ± 1 when $h = g$ and is 0 when $h \neq g$. Here $\mu^2 : hom(\mathcal{P}, L_1) \times hom(\tau_P(L_1'), \mathcal{P}) \to hom(\tau_P(L_1'), L_1)$ is the multiplication (Fig. 9).*

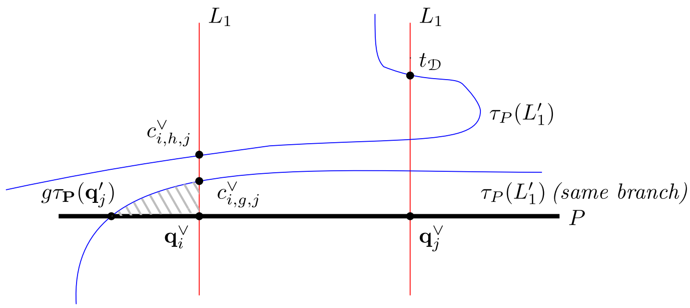

Fig. 9 Holomorphic triangles in $\mathbf{U}$

Proof First, we want to argue that any $u \in \mathcal{M}^{J^\tau}(c^\vee_{i,h,j}; q_i^\vee, \tau_P(q_j'))$ contributing to $\mu^2(\mathbf{q}_i^\vee, g\tau_P(\mathbf{q}_j'))$ has image completely lying inside U when $\tau \gg 1$. We argue as before. Let $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ be a limiting holomorphic building. By boundary condition, there is $v_1 \in V(\mathcal{T})$ such that $q_i^\vee$, $\tau_P(q_j')$ are asymptotes of u_{v_1}. If $c^\vee_{i,h,j}$ is not an asymptote of u_{v_1}, then we get a contradiction as in Lemma 5.11. Therefore, u_{v_1} has asymptotes $c^\vee_{i,h,j}$, $q_i^\vee$, $\tau_P(q_j')$ and positive Reeb chords $y_1, \dots, y_k$. The virtual dimension of u_{v_1} is given by

$$\mathrm{virdim}(u_{v_1}) = |c^\vee_{i,h,j}| - |q_i^\vee| - |\tau_P(q_j')| - \sum_{s=1}^{k} |y_s| + k \geqslant 1 - 0 - 1 + k = k$$

It means that $k = 0$ so u_{v_1} has no positive Reeb chord and the claim follows.

In particular, we can lift $u \in \mathcal{M}^{J^\tau}(c^\vee_{i,h,j}; q_i^\vee, \tau_P(q_j'))$ to the universal cover $\mathbf{U}$. By considering the boundary condition, it is clear that we must have $h = g$ for u to exist. Now, to compute the $c^\vee_{i,g,j}$-coefficient of $\mu^2(\mathbf{q}_i^\vee, g\tau_P(\mathbf{q}_j'))$, we use the following model.

We consider an A_3-Milnor fiber as in the proof of Theorem 4.14 but rename the objects to keep the notation aligned with the current situation. For example, we denote the Lagrangian spheres by S_1 ,S and S_2 such that $S_1 \cap S_2 = \varnothing$. Let τ be the Dehn twist along S, $q^\vee \in CF(S, S_1)$, $q' \in CF(S_2, S)$, $\tau(q') \in CF(\tau(S_2), S)$, $c^\vee \in CF(\tau(S_2), S_1)$ and $e, f \in CF(S, S)$. We have $|q^\vee| = 0$, $|q'| = n$, $|\tau(q')| = 1$, $|c^\vee| = 1$, $|e| = 0$ and $|f| = n$. Consider the following commutative diagram (up to sign)

$$\begin{tikzcd}
HF(\tau(S_2), S) \times HF(S_1, \tau(S_2)) \times HF(S, S_1) \arrow[r, "Id \times \mu^2"] \arrow[d, "\mu^2(\tau(q'), \cdot) \times Id"] & HF(\tau(S_2), S) \times HF(S, \tau(S_2)) \arrow[d, "\mu^2(\tau(p'), \cdot)"] \\
HF(S_1, S) \times HF(S, S_1) \arrow[r, "\mu^2"] & HF(S, S)
\end{tikzcd}$$

All the Floer cohomology has rank 1 except that $HF(S, S)$ has rank 2. The bottom arrow gives $\mu^2(q, q^\vee) = f$. By the long exact sequence

$$HF^k(S_1, S_2) \to HF^k(S_1, \tau(S_2)) \to HF^{k+1}(S_1, S) \to HF^{k+1}(S_1, S_2) \quad (5.16)$$

and the fact that $HF(S_1, S_2) = 0$, we know that $HF^{n-1}(S_1, \tau(S_2)) \to HF^n(S_1, S)$ is an isomorphism. Since $\tau(q')$ represents the unique (up to multiplications by a unit) non-zero class in $HF(\tau(S_2), S)$, we know that $\mu^2(\tau(q'), \cdot)$ induces the isomorphism $HF^{n-1}(S_1, \tau(S_2)) \simeq HF^n(S_1, S)$. Therefore, we must have $\mu^2(\tau(q'), c) = \pm q$.

By the associativity of cohomological multiplication, we have $\mu^2(\tau(q'), \mu^2(c, q^\vee)) = \pm f$. It implies that $\mu^2(c, q^\vee) = \pm\tau(q')^\vee$. Dually, we have $\mu^2(q^\vee, \tau(q')) = \pm c^\vee$ (it amounts to changing the asymptote c from outgoing end to incoming end, and $\tau(q')$ from incoming end to outgoing end).

Since each $u \in \mathcal{M}^{J^\tau}(c^\vee_{i,h,j}; q_i^\vee, \tau_P(q_j'))$ can be lifted to $\mathbf{U}$, there is a sign preserving bijective correspondence $\mathcal{M}^{J^\tau}(c^\vee_{i,h,j}; q_i^\vee, \tau_P(q_j')) \simeq \mathcal{M}(c^\vee; q^\vee, \tau(q'))$ so we get the result. □

Remark 5.13 There is an alternative geometric argument as follows. When the fibers corresponding S_1 and S_2 in the proof of Lemma 5.12 are fibers of antipodal points. The moduli computing $c^\vee$-coefficient of $\mu^2(q^\vee, \tau(q'))$ is the constant map to the point $S_1 \cap S$. One can check that this constant map is regular so the algebraic count is ± 1. In the more general case, where $S \cap S_2$ is not the antipodal point of $S_1 \cap S$, one can apply a homotopy type argument to conclude Lemma 5.12.

Now we enrich the statement of Lemma 5.12 by adding the local system on L_1 and L_1' into consideration. Take the universal cover $\mathbf{U}$ of the neighborhood of P, there is a unique path (up to homotopy) in $\tau_{\mathbf{P}}(T^*_{g\mathbf{q}_j'}\mathbf{P})$ from $\mathbf{c}_{\mathbf{q}_i, g\mathbf{q}_j'}$ to $g\tau_{\mathbf{P}}(\mathbf{q}_j')$. It descends to the unique path (up to homotopy) in $\tau_P(T^*_{q_j'}P)$ from $c_{\mathbf{q}_i, g\mathbf{q}_j'}$ to $\tau_P(q_j')$, which we denote by $[c_{\mathbf{q}_i, g\mathbf{q}_j'} \to \tau_P(q_j')]$. Similarly, there is a unique path (up to homotopy) in $T^*_{q_i}P$ from q_i to $c_{\mathbf{q}_i, g\mathbf{q}_j'}$, which we denote by $[q_i \to c_{\mathbf{q}_i, g\mathbf{q}_j'}]$. Then we have

Proposition 5.14 *For $\tau \gg 1$, we have [see (5.6)], up to sign,*

$$\mu^2(\psi^2 \otimes \mathbf{q}_i^\vee, g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1) = I_{[q_i \to c_{\mathbf{q}_i, g\mathbf{q}_j'}]}(\psi^2) \otimes (\psi^1 \circ I_{[c_{\mathbf{q}_i, g\mathbf{q}_j'} \to \tau_P(q_j')]}) \in Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{c_{\mathbf{q}_i, g\mathbf{q}_j'}}, \mathcal{E}^1_{c_{\mathbf{q}_i, g\mathbf{q}_j'}}) \tag{5.17}$$

for $\psi^2 \in \mathcal{E}^1_{q_i}$ and $\psi^1 \in Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{\tau_P(q_j')}, \mathbb{K})$. In particular, the right hand side is supported at the intersection point $c^\vee_{i,g,j}$ only and the morphism $\Phi^\otimes_{i,g,j} := \mu^2(- \otimes \mathbf{q}_i^\vee, g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes -)$

$$\Phi^\otimes_{i,g,j} : \mathcal{E}^1_{q_i} \otimes Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{\tau_P(q_j')}, \mathbb{K}) \to Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{c_{\mathbf{q}_i, g\mathbf{q}_j'}}, \mathcal{E}^1_{c_{\mathbf{q}_i, g\mathbf{q}_j'}}) \tag{5.18}$$

is an isomorphism.

Note that the parallel transport from $\tau_P(q_j')$ to q_i in the statement was omitted for a reason that will become clear from the proof.

Proof By Lemmas 5.10, 5.11 and 5.12, we already know that $\mu^2(\psi^2 \otimes \mathbf{q}_i^\vee, g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1)$ is supported at the intersection point $c_{i,g,j}^\vee$. Moreover, as explained in the proof of Lemma 5.12, the rigid elements contributing to $\mu^2(\mathbf{q}_i^\vee, g\tau_P(\mathbf{q}_j'))$ lie completely inside U.

To obtain the result, it suffices to understand the parallel transport maps. Let $u \in \mathcal{M}^{J^\tau}(c_{i,g,j}^\vee; q_i^\vee, \tau_P(q_j'))$. The contribution to $\mu^2(\psi^2 \otimes \mathbf{q}_i^\vee, g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1)$ by u is given by (up to sign)

$$(I_{\partial_2 u} \circ \psi^2) \otimes (\mathbf{q}_i^\vee \circ I_{\partial_1 u} \circ g\tau_{\mathbf{P}}(\mathbf{q}_j')) \otimes (\psi^1 \circ I_{\partial_0 u}) \tag{5.19}$$

Since the domain of u is contractible, u can be lifted to the universal cover and therefore the generator $c_{i,g,j}^\vee$ uniquely determine the homotopy class of the path $\partial_1 u$ on P (and also $\partial_0 u$ on $\tau_P(L_1')$ and $\partial_2 u$ on L_1, which is why the parallel transport of $\partial_1 u$ is omitted in the statement), which is exactly the path such that $\mathbf{q}_i^\vee \circ I_{\partial_1 u} \circ g\tau_{\mathbf{P}}(\mathbf{q}_j') = 1$, where $g\tau_{\mathbf{P}}(\mathbf{q}_j')$ is regarded as an element of the universal local system at q_j' and $\mathbf{q}_i^\vee$ is regarded as an element of the dual of the universal local system at q_i. In other words, we have $I_{\partial_1 u}(g\tau_{\mathbf{P}}(\mathbf{q}_j')) = \mathbf{q}_i$. On the other hand, we have $I_{\partial_0 u} = I_{[c_{\mathbf{q}_i, g\mathbf{q}_j'} \to \tau_P(q_j')]}$ and $I_{\partial_2 u} = I_{[q_i \to c_{\mathbf{q}_i, g\mathbf{q}_j'}]}$ so (5.19) reduces to $I_{[q_i \to c_{\mathbf{q}_i, g\mathbf{q}_j'}]}(\psi^2) \otimes (\psi^1 \circ I_{[c_{\mathbf{q}_i, g\mathbf{q}_j'} \to \tau_P(q_j')]})$. Now, (5.17) follows immediately from Lemma 5.12.

On the other hand, since $I_{[q_i \to c_{\mathbf{q}_i, g\mathbf{q}_j'}]}$ and $I_{[c_{\mathbf{q}_i, g\mathbf{q}_j'} \to \tau_P(q_j')]}$ are isomorphisms from $\mathcal{E}^1_{q_i}$ to $\mathcal{E}^1_{c_{\mathbf{q}_i, g\mathbf{q}_j'}}$ and from $\tau_P((\mathcal{E}^1)')_{c_{\mathbf{q}_i, g\mathbf{q}_j'}}$ to $\tau_P((\mathcal{E}^1)')_{\tau_P(q_j')}$, respectively, (5.17) clear induces the isomorphism

$$\mathcal{E}^1_{q_i} \otimes Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{\tau_P(q_j')}, \mathbb{K}) \to \mathcal{E}^1_{c_{\mathbf{q}_i, g\mathbf{q}_j'}} \otimes Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{c_{\mathbf{q}_i, g\mathbf{q}_j'}}, \mathbb{K}) \tag{5.20}$$

as desired □

With these preparation, we go back to the study of the degree zero cocycles of $\mathcal{D}$.

Corollary 5.15 *For L_1' sufficiently close to L_1 and $\tau \gg 1$, every degree 0 class in $H^0(\mathcal{D})$ admits a cochain representative β which is a sum of elements supported at e_L and $\{q_i^\vee \otimes \tau(q_j')\}_{i,j}$ only. Moreover, the term of β supported at e_L cannot be zero unless $\beta = 0$.*

Proof Every degree 0 cocycle in $\mathcal{D}$ is a sum of elements supported at e_L, $\{c_{i,g,j}^\vee\}_{i,j,g}$ and $\{q_i^\vee \otimes \tau(q_j')\}_{i,j}$ because $|w_k| \neq 0$ for $w_k \neq e_L$. Let β be a degree 0 cocycle which represents a class $[\beta]$. By Proposition 5.14, we can eliminate the terms of β supported at $c_{i,g,j}^\vee$ by adding the $\mu^1_{\mathcal{D}}$-differentials of certain cochains supported at $q_i^\vee \otimes \tau(q_j')$. Note that the term of β supported at $c_{i,g,j}^\vee$ themselves might not be exact because $\mu^1((\psi^2 \otimes \mathbf{q}_i^\vee) \otimes (g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1))$ involves more than just $\mu^2_{\mathcal{F}}(\psi^2 \otimes \mathbf{q}_i^\vee, g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1)$ [see (5.9)], but the remainder terms cannot have $c_{i,g,j}^\vee$-components.

Therefore, we have a cochain β' cohomologous to β such that β' is a sum of elements supported at e_L and $\{q_i^\vee \otimes \tau(q_j')\}_{i,j}$ only.

Now, suppose the term of β' supported at e_L is 0. We write $\beta' = \sum_{(i,j)} \psi^{i,j}$, where, for all i, j, $\psi^{i,j}$ is an element supported at $q_i^\vee \otimes \tau_P(q_j')$. If $\psi^{i_0,j_0} \neq 0$ for some i_0, j_0, then by the isomorphism statement in Proposition 5.14, the terms of $\mu^1(\beta')$ must contain a non-trivial element supported at $c^\vee_{i_0,g,j_0}$ for some g. Because all other $\mu^1(\psi^{i,j})$ do not have non-zero element supported at $c^\vee_{i_0,g,j_0}$, this draws a contradiction. □

By Corollary 5.15, we can write every degree 0 cocycle β of $\mathcal{D}$ as

$$\beta = \psi_{e_L} + \sum_{i,j} \psi_{q_i^\vee \otimes \tau_P(q_j')} \tag{5.21}$$

where ψ_x is an element supported at x. Moreover, by (5.6), we can further decompose $\psi_{q_i^\vee \otimes \tau_P(q_j')}$ as

$$\psi_{q_i^\vee \otimes \tau_P(q_j')} = \sum_{g\in\Gamma} \sum_{k=1}^{n_{i,g,j}} \psi^2_{i,g,j,k} \otimes \mathbf{q}_i^\vee \otimes g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1_{i,g,j,k} \tag{5.22}$$

for some $\psi^2_{i,g,j,k} \in \mathcal{E}^1_{q_i}$, $\psi^1_{i,g,j,k} \in Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{\tau_P(q_j')}, \mathbb{K})$ and $n_{i,g,j} \in \mathbb{N}$.

Proposition 5.16 (Cocycle elements) *For L_1' sufficiently close to L_1 and $\tau \gg 1$, there is a non-exact degree 0 cocycle $c_{\mathcal{D}}$ in $\mathcal{D}$ of the form*

$$c_{\mathcal{D}} = t_{\mathcal{D}} + \sum_{g,k,i,j} \psi^2_{i,g,j,k} \otimes \mathbf{q}_i^\vee \otimes g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1_{i,g,j,k} \tag{5.23}$$

where $\psi^2_{i,g,j,k} = \psi^1_{i,g,j,k} = 0$ if either $j > i$ or ($j = i$ and $g \neq 1_\Gamma$), and [*see* (5.18)]

$$\Phi^{\otimes}_{i,1_\Gamma,i}\left(\sum_k (\psi^2_{i,1_\Gamma,i,k} \otimes \mathbf{q}_i^\vee \otimes \tau_{\mathbf{P}}(\mathbf{q}_i') \otimes \psi^1_{i,1_\Gamma,i,k})\right) = \pm id \tag{5.24}$$

where $\pm id \in Hom_{\mathbb{K}}(\tau_P((\mathcal{E}^1)')_{c_{\mathbf{q}_i,\mathbf{q}_i'}}, \mathcal{E}^1_{c_{\mathbf{q}_i,\mathbf{q}_i'}})$.

Proof Let β be a non-exact degree 0 cocycle of $\mathcal{D}$ [which exists from (5.2)]. We write β in the form (5.21). Note that ψ_{e_L} can be geometrically identified as an element of $hom((\mathcal{E}^1)', \mathcal{E}^1)$. Lemma 5.4 implies that, for $\mu^1_{\mathcal{D}}(\beta) = 0$, we must have $\mu^1_{hom((\mathcal{E}^1)',\mathcal{E}^1)}(\psi_{e_L}) = 0$.

Also, Corollary 5.15 implies that the degree zero cocycle β is uniquely determined by its ψ_{e_L} component (or, as a cochain of $\mathcal{D}$, $\mu^1_{\mathcal{D}}(\psi_{e_L})$ has no w_k-components). Therefore,

$$\operatorname{rank}(H^0(\mathcal{D})) \leqslant \operatorname{rank}(H^0(hom((\mathcal{E}^1)', \mathcal{E}^1))) \tag{5.25}$$

However, as explained in (5.2), we have

$$\operatorname{rank}(H^0(\mathcal{D})) = \operatorname{rank}(HF^0((\mathcal{E}^1)', \mathcal{E}^1)) \tag{5.26}$$

It implies that for each degree 0 cocycle $\psi_{e_L} \in hom((\mathcal{E}^1)', \mathcal{E}^1)$, there exists $\psi_{q_i^\vee \otimes \tau_P(q_j')}$ such that $\psi_{e_L} + \sum_{i,j} \psi_{q_i^\vee \otimes \tau_P(q_j')}$ is a degree 0 cocycle in $\mathcal{D}$.

In particular, we can take $\psi_{e_L} = t_{\mathcal{D}}$. For $\mu^1(t_{\mathcal{D}} + \sum_{i,j} \psi_{q_i^\vee \otimes \tau_P(q_j')}))$ to be zero, the terms of it supported at $c_{i,g,j}^\vee$ must be zero for all i, j, g. Therefore, we obtain the result by Propositions 5.8 and 5.14 [see (5.9)]. □

5.2 Quasi-isomorphisms

Let $c_{\mathcal{D}}$ be the degree 0 cocycle obtained from Proposition 5.16. In this section, we are going to study the map (5.1) for $\mathcal{E}^0 \in Ob(\mathcal{F})$.

We assume that $L_0 \pitchfork L_1$, $L_0 \pitchfork \tau_P(L_1')$, and that $L_0 \cap U$ is a union of cotangent fibers $\bigcup_{i=1}^{d_{L_0}} T_{p_i}^* P$, where $d_{L_0} = \#(L_0 \cap P)$. Let $\mathbf{p}_i$ be a choice of lift of p_i in $\mathbf{P}$. Let $C_0 := hom(\mathcal{E}^0, \tau_P((\mathcal{E}^1)'))$ and $C_1 := hom(\mathcal{E}^0, T_{\mathcal{P}}(\mathcal{E}^1))$. We know from Lemma 4.13 that, when τ is large enough, there is a subcomplex $C_0^s \subset C_0$ generated by generators of C_0 outside U. Let $C_0^q := C_0/C_0^s$ be the quotient complex, which is generated by generators of C_0 inside U. Similarly, $C_1^s := hom(\mathcal{E}^0, \mathcal{E}^1) \subset C_1$ is a subcomplex and $C_1^q := C_1/C_1^s$ is the quotient complex. By definition [see (2.88)], for $\psi \in C_0$,

$$\begin{aligned}\mu^2(c_{\mathcal{D}}, \psi) &= \mu^2_{\mathcal{F}}(t_{\mathcal{D}}, \psi) + \sum_{i,j,g,k} (\psi^2_{i,g,j,k} \otimes \mathbf{q}_i^\vee) \otimes \mu^2_{\mathcal{F}}(g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1_{i,g,j,k}, \psi) \\ &\quad + \sum_{i,j,g,k} \mu^3_{\mathcal{F}}(\psi^2_{i,g,j,k} \otimes \mathbf{q}_i^\vee, g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1_{i,g,j,k}, \psi) \end{aligned} \tag{5.27}$$

We define $\mu_s^2(c_{\mathcal{D}}, -) := \mu^2(c_{\mathcal{D}}, -)|_{C_0^s} : C_0^s \to C_1$.

Lemma 5.17 *For $\tau \gg 1$, the image of $\mu_s^2(c_{\mathcal{D}}, -)$ is contained in C_1^s. Therefore, $\mu_s^2(c_{\mathcal{D}}, -) : C_0^s \to C_1^s$ is a chain map.*

Proof Note that the first and last term on the right hand side of (5.27) lie inside C_1^s as a consequence of Lemma 4.5. Therefore, it suffices to show that $\mu^2_{\mathcal{F}}(g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi^1_{i,g,j,k}, \psi) = 0$ for $\psi \in C_0^s$. We consider the moduli $\mathcal{M}^{J^\tau}(p_s; \tau_P(q_j'), y)$ where $y \in (L_0 \cap \tau_P(L_1'))\backslash U$ and $p_s \in L_0 \cap P$. Let $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ be a holomorphic building converging from curves in $\mathcal{M}^{J^\tau}(p_s; \tau_P(q_j'), y)$. From the boundary condition, there exists $v_1 \in V(\mathcal{T})$ such that p_s and $\tau_P(q_j')$ are asymptotes of u_{v_1}. The other asymptotes of u_{v_1} are positive Reeb chords $y_1, \ldots, y_m$. We have

$$\mathrm{virdim}(u_{v_1}) = |p_s| - |\tau_P(q_j')| - \sum_{l=1}^m |y_l| - (1-m) \geqslant n - 1 - (1-m) \geqslant n - 2 > 0, \tag{5.28}$$

contradiction. Therefore, $\mathcal{M}^{J^\tau}(p_s; \tau_P(q_j'), y) = \varnothing$ for $\tau \gg 1$. □

Lemma 5.18 *For $\tau \gg 1$, $\mu_s^2(c_{\mathcal{D}}, -) = \mu^2_{\mathcal{F}}(t_{\mathcal{D}}, -)$.*

Proof By Lemma 5.17, the second term in (5.27) vanishes, so it suffices to prove that $\mathcal{M}^{J^\tau}(x; q_i^\vee, \tau_P(q_j'), y) = \varnothing$ for $\tau \gg 1$, where $y \in (L_0 \cap \tau_P(L_1'))\backslash U$ and $x \in L_0 \cap L_1$. Let $u_\infty = (u_v)_{v \in V(\mathcal{T})}$ be a holomorphic building converging from curves in $\mathcal{M}^{J^\tau}(x; q_i^\vee, \tau_P(q_j'), y)$. From the boundary condition, there exists $v_1 \in V(\mathcal{T})$ such that $q_i^\vee$ and $\tau_P(q_j')$ are asymptotes of u_{v_1}. The other asymptotes of u_{v_1} are positive Reeb chords $y_1, \ldots, y_m$. We have

$$\begin{aligned} \operatorname{virdim}(u_{v_1}) =& n - |q_i^\vee| - |\tau_P(q_j')| \\ & - \sum_{l=1}^{m} |y_l| - (1-m) \geqslant n - 1 - (1-m) \geqslant n - 2 > 0 \end{aligned} \tag{5.29}$$

Therefore, $\mathcal{M}^{J^\tau}(x; q_i^\vee, \tau_P(q_j'), y) = \varnothing$ for $\tau \gg 1$, □

Proposition 5.19 *For $\tau \gg 1$, $\mu_s^2(c_{\mathcal{D}}, -)$ is a quasi-isomorphism.*

Proof For $y \in (L_0 \cap \tau_P(L_1'))\backslash U$ and $x \in L_0 \cap L_1$, the proof of Lemma 4.8 implies that all rigid elements in $\mathcal{M}^{J^\tau}(x; e_L, y)$ have their image completely outside U.

As a result, the computation of $\mu_s^2(c_{\mathcal{D}}, -) = \mu_{\mathcal{F}}^2(t_{\mathcal{D}}, -)$ picks up exactly the same holomorphic triangles that contributes to $\mu_{\mathcal{F}}^2(e_{\mathcal{E}}, -) : C_0^s \cong hom(\mathcal{E}^0, (\mathcal{E}^1)') \to hom(\mathcal{E}^0, \mathcal{E}^1) \cong C_1^s$ via the tautological identification between $e_{\mathcal{E}}$ and $t_{\mathcal{D}}$ [see (5.12) and the paragraph after it]. Since $e_{\mathcal{E}}$ is the cohomological unit, $\mu_s^2(c_{\mathcal{D}}, -)$ is also a quasi-isomorphism. □

By Lemma 5.17, we know that $\mu^2(c_{\mathcal{D}}, -)$ induces a chain map on the quotient complexes $\mu_q^2(c_{\mathcal{D}}, -) : C_0^q \to C_1^q$. Since the first and last term on the right hand side of (5.27) are, by definition, lying inside C_1^s, the map $\mu_q^2(c_{\mathcal{D}}, -)$ is given by

$$\mu_q^2(c_{\mathcal{D}}, \psi) = \sum_{i,j,g,k} (\psi_{i,g,j,k}^2 \otimes \mathbf{q}_i^\vee) \otimes \mu_{\mathcal{F}}^2(g\tau_{\mathbf{P}}(\mathbf{q}_j') \otimes \psi_{i,g,j,k}^1, \psi) \tag{5.30}$$

By Proposition 5.19 and the five lemma, to show that $\mu^2(c_{\mathcal{D}}, -)$ is a quasi-isomorphism, it suffices to show that $\mu_q^2(c_{\mathcal{D}}, -)$ is a quasi-isomorphism.

We recall from Lemma 4.2 that there is a bijective correspondence

$$\iota : hom(\mathcal{P}, L_1') \otimes_\Gamma hom(L_0, \mathcal{P}) \to (L_0 \cap \tau_P(L_1')) \cap U \tag{5.31}$$

so we can write a point $y \in (L_0 \cap \tau_P(L_1')) \cap U$ as $c_{h\mathbf{p}_s, \mathbf{q}_l'} := \iota(\mathbf{q}_l'^\vee \otimes h\mathbf{p}_s)$ for some $h \in \Gamma$ and some s, l. We want to understand the moduli $\mathcal{M}^{J^\tau}(p_m; \tau_P(q_j'), c_{h\mathbf{p}_s, \mathbf{q}_l'})$ for various j, s, l, m, which is responsible for (part of) the operation

$$hom(\tau_P(L_1'), \mathcal{P}) \times hom(L_0, \tau_P(L_1')) \to hom(L_0, \mathcal{P}) \tag{5.32}$$

Notice that, by switching the appropriate strip-like ends from incoming to outgoing (and vice versa) for the same holomorphic triangles, (5.32) can be dualized to

$$hom(\mathcal{P}, L_0) \times hom(\tau_P(L'_1), \mathcal{P}) \to hom(\tau_P(L'_1), L_0) \tag{5.33}$$

If we replace L_0 by L_1 (both of them are union of cotangent fibers in U), then we see that (5.33) has already been studied in Lemmas 5.11 and 5.12. The outcome is the following.

Lemma 5.20 *For $\tau \gg 1$, for $\psi_{c_{h\mathbf{p}_s,\mathbf{q}'_l}} \in C_0$ supported at $c_{h\mathbf{p}_s,\mathbf{q}'_l}$*

$$\mu^2_{\mathcal{F}}(g\tau_{\mathbf{P}}(\mathbf{q}'_j) \otimes \psi^1_{i,g,j,k}, \psi_{c_{h\mathbf{p}_s,\mathbf{q}'_l}}) \tag{5.34}$$

is 0 *if $l \neq j$. When $l = j$,* (5.34) *becomes*

$$gh\mathbf{p}_s \otimes (I_{[\tau_P(q'_j)\to p_s]} \circ \psi^1_{i,g,j,k} \circ I_{[c_{h\mathbf{p}_s,\mathbf{q}'_j} \to \tau_P(q'_j)]} \circ \psi_{c_{h\mathbf{p}_s,\mathbf{q}'_j}} \circ I_{[p_s \to c_{h\mathbf{p}_s,\mathbf{q}'_j}]}) \tag{5.35}$$

where all the parallel transport maps are the unique one determined by the boundary condition inside U (cf. Proposition 5.14*).*

Proof The argument largely resembles the proof of Lemmas 5.11, 5.12 and Proposition 5.14. A neck-stretching argument as in Lemma 5.11 deduces that $\mathcal{M}^{J^\tau}(p_m; \tau_P(q'_j), c_{h\mathbf{p}_s,\mathbf{q}'_l})$ is not empty only if $j = l$ and $m = s$. The same dimension count implies that when $j = l$, $m = s$ and $\tau \gg 1$, every rigid element of $\mathcal{M}^{J^\tau}(p_m; \tau_P(q'_j), c_{h\mathbf{p}_s,\mathbf{q}'_l})$ has image inside U. The local count and the chasing of local systems from Lemma 5.12 and Proposition 5.14 applies directly to the current case because it is a computation in $\mathbf{U}$ about cotangent fibers and their Dehn twists. In particular, if we remove the local systems on L_0 and $\tau_P(L'_1)$, we get

$$\mu^2_{\mathcal{F}}(g\tau_{\mathbf{P}}(\mathbf{q}'_j), c_{h\mathbf{p}_s,\mathbf{q}'_j}) = \mu^2_{\mathcal{F}}(g\tau_{\mathbf{P}}(\mathbf{q}'_j), c_{gh\mathbf{p}_s,g\mathbf{q}'_j}) = gh\mathbf{p}_s \in hom(L_0, \mathcal{P}) \tag{5.36}$$

The parallel transport maps are uniquely determined by boundary conditions, and after chasing all of them, we get the result. □

Let $V = hom(\mathcal{P}, (\mathcal{E}^1)') \otimes_\Gamma hom(\mathcal{E}^0, \mathcal{P})$ which is generated by elements of the form

$$(\Upsilon^2 \otimes (\mathbf{q}'_r)^\vee) \otimes (h\mathbf{p}_t \otimes \Upsilon^1) \tag{5.37}$$

for $h \in \Gamma$, $r = 1, \dots, d_{L_1}$, $t = 1, \dots, d_{L_0}$, $\Upsilon^2 \in (\mathcal{E}^1)'_{q'_r}$ and $\Upsilon^1 \in Hom_{\mathbb{K}}(\mathcal{E}^0_{p_t}, \mathbb{K})$ [cf. (5.6)].

- For $s = 1, \dots, d_{L_0}$, let V_s be the subspace generated by elements in (5.37) such that $t = s$.
- For $s = 1, \dots, d_{L_0}$ and $l = 1, \dots, d_{L_1}$, let $V_{s,l}$ be the subspace of V_s generated by elements in (5.37) such that $r = l$.
- For $s = 1, \dots, d_{L_0}$, $l = 1, \dots, d_{L_1}$ and $g \in \Gamma$, let $V_{s,l,g}$ be the subspace of $V_{s,l}$ generated by elements in (5.37) such that $h = g$.

Therefore, we have direct sum decompositions

$$V = \oplus_s V_s, \;\; V_s = \oplus_l V_{s,l}, \;\; V_{s,l} = \oplus_g V_{s,l,g} \tag{5.38}$$

The bijective correspondence ι (5.31) extends to an isomorphism, also denoted by ι, from V to C_0^q by keeping track of the (uniquely determined) parallel transport maps along Lagrangians inside U. On the other hand, there is an obvious isomorphism $F : hom(\mathcal{P}, \mathcal{E}^1) \otimes_\Gamma hom(\mathcal{E}^0, \mathcal{P}) \to V$ given by

$$(\Upsilon^2 \otimes \mathbf{q}_l^\vee) \otimes (h\mathbf{p}_s \otimes \Upsilon^1) \mapsto (\Upsilon^2 \otimes (\mathbf{q}_l')^\vee) \otimes (h\mathbf{p}_s \otimes \Upsilon^1) \tag{5.39}$$

where we used the identification $\mathcal{E}^1_{q_l} \simeq (\mathcal{E}^1)'_{q_l}$ by the Hamiltonian push-off. As a result, we have a composition map

$$\Theta : V \xrightarrow{\iota} (L_0 \cap \tau_P(L_1')) \cap U \xrightarrow{\mu_q^2(c_{\mathcal{D}}, -)} C_1^q \xrightarrow{F} V \tag{5.40}$$

which respects a filtration on V in the following sense.

Lemma 5.21 *We have*

$$\begin{cases} \Theta(V_s) \subset V_s & \text{for all } s \\ \Theta(V_{s,l}) \subset \bigoplus_{t \geqslant l} V_{s,t} & \text{for all } s, l \\ \Theta(V_{s,l,h}) \subset V_{s,l,h} + (\bigoplus_{t > l} V_{s,t}) & \text{for all } s, l, h \end{cases} \tag{5.41}$$

Proof Explicitly, Θ is given by [see (5.30) and Lemma 5.20]

$$(\Upsilon^2 \otimes (\mathbf{q}_l')^\vee) \otimes (h\mathbf{p}_s \otimes \Upsilon^1) \tag{5.42}$$

$$\mapsto \sum_{i,g,k} (\psi^2_{i,g,l,k} \otimes (\mathbf{q}_i')^\vee) \otimes (gh\mathbf{p}_s \otimes R(\psi^1_{i,g,l,k}, \Upsilon^2, \Upsilon^1)) \tag{5.43}$$

where $R(\psi^1_{i,g,l,k}, \Upsilon^2, \Upsilon^1)$ is a term depending on $\psi^1_{i,g,l,k}, \Upsilon^2, \Upsilon^1$ given by composing parallel transport maps. It is therefore clear that $\Theta(V_s) \subset V_s$. By Propositions 5.16 and (5.27), we know that $\psi^2_{i,g,l,k} = 0$ unless $j \leqslant i$ so $\Theta(V_{s,l}) \subset \oplus_{t \geqslant l} V_{s,t}$.

When $i = l$, $\psi^2_{i,g,l,k} \neq 0$ only if $g = 1_\Gamma$ (by Proposition 5.16). Therefore, $\Theta(V_{s,l,h}) \subset V_{s,l,h} + (\oplus_{t>l} V_{s,t})$ □

Proposition 5.22 *μ_q^2 is a quasi-isomorphism.*

Proof Since μ_q^2 is a chain map, it suffices to show that μ_q^2 is bijective. We know that ι and F are isomorphisms so it suffices to show that Θ is surjective [see (5.40)]. By (5.38) and Lemma 5.21, it suffices to show that

$$\Theta|_{V_{s,l,h}} : V_{s,l,h} \to (V_{s,l,h} + (\oplus_{t>l} V_{s,t}))/(\oplus_{t>l} V_{s,t}) \tag{5.44}$$

is bijective for all s, l, h. For fixed s, l, h, the map (5.44) can be identified with the map

$$(\mathcal{E}^1)'_{q'_l} \otimes Hom_{\mathbb{K}}(\mathcal{E}^0_{p_s}, \mathbb{K}) \to (\mathcal{E}^1)'_{q'_l} \otimes Hom_{\mathbb{K}}(\mathcal{E}^0_{p_s}, \mathbb{K})$$
$$\Upsilon^2 \otimes \Upsilon^1 \mapsto \sum_k (\psi^2_{l,1_\Gamma,l,k} \otimes R(\psi^1_{l,1_\Gamma,l,k}, \Upsilon^2, \Upsilon^1)) \tag{5.45}$$

By (5.24) and keeping track of the uniquely determined parallel transport maps, it is clear that (5.45) is an isomorphism. □

Concluding the proof of Theorem 1.2, 5.1 For each $\mathcal{E}^1 \in Ob(\mathcal{F})$, we apply Proposition 5.16 to find a degree 0 cocycle $c_{\mathcal{D}} \in hom^0_{\mathcal{F}^{perf}}(\tau_P((\mathcal{E}^1)'), T_{\mathcal{P}}(\mathcal{E}^1))$. Given any object $(\mathcal{E}^0)' \in Ob(\mathcal{F})$, we consider a quasi-isomorphic $\mathcal{E}^0$, which is a Hamiltonian isotopic copy and the underlying Lagrangian L_0 intersects transversally with L_1, $\tau_P(L'_1)$ and $L_0 \cap U$.

Propositions 5.19 and 5.22, together with the five lemma, then conclude that (5.1) is a quasi-isomorphism. □

Proof of Corollary 1.3 When P is diffeomorphic to $\mathbb{RP}^n$ and $n = 4k - 1$, P is spin and can be equipped with the spin structure descended from S^n. When $\mathrm{char}(\mathbb{K}) \neq 2$, the universal local system $\mathcal{P}$ is a direct sum of two rank 1 local systems $\mathcal{E}^1$ and $\mathcal{E}^2$. This is because $\mathbb{K}[\mathbb{Z}_2]$ splits when $\mathrm{char}(\mathbb{K}) \neq 2$. Moreover, by Lemma 2.10 and Corollary 2.11, we have

$$HF^*(\mathcal{E}^i, \mathcal{E}^j) = \begin{cases} 0 & \text{if } i \neq j \\ H^*(S^n) & \text{if } i = j \end{cases} \tag{5.46}$$

so $\mathcal{E}^1$ and $\mathcal{E}^2$ are orthogonal spherical objects. In this case,

$$T_{\mathcal{P}}(\mathcal{E}) \simeq Cone(\oplus_{i=1,2}(hom_{\mathcal{F}}(\mathcal{E}^i, \mathcal{E}) \otimes \mathcal{E}^i) \xrightarrow{ev} \mathcal{E}) \tag{5.47}$$

where ev is the evaluation map. The spherical twist to $\mathcal{E}$ along $\mathcal{E}^i$ is defined to be $Cone(hom_{\mathcal{F}}(\mathcal{E}^i, \mathcal{E}) \otimes \mathcal{E}^i \xrightarrow{ev} \mathcal{E})$. A direct verification shows that (5.47) is the same as applying the spherical twist to $\mathcal{E}$ along $\mathcal{E}^1$ and then $\mathcal{E}^2$. It is the same as first applying spherical twist along $\mathcal{E}^2$ and then $\mathcal{E}^1$ because $\mathcal{E}^1$ and $\mathcal{E}^2$ are orthogonal objects.

When P is diffeomorphic to $\mathbb{RP}^n$ and $\mathrm{char}(\mathbb{K}) = 2$, then $H^*(P) = H^*(\mathbb{RP}^n, \mathbb{Z}_2)$. In this case, one can define a $\mathbb{P}$-twist along P (see [5,47]) which is an auto-equivalence on $\mathcal{F}^{perf}$. More precisely, the algebra $H^*(\mathbb{RP}^n, \mathbb{Z}_2)$ is generated by a degree 1 element instead of a degree 2 element so the $\mathbb{P}$-twist along P is not exactly, but a simple variant of, the $\mathbb{P}$-twist defined in [5]. To compare (1.1) with the $\mathbb{P}$-twist, we note that $\mathbb{K}[\mathbb{Z}_2]$ fits into a non-split exact sequence

$$0 \to \mathbb{K} \to \mathbb{K}[\mathbb{Z}_2] \to \mathbb{K} \to 0 \tag{5.48}$$

it implies that $\mathcal{P} = Cone(P[-1] \to P)$ and the morphism in the cone is the unique non-trivial one. In this case, the fact that $T_{\mathcal{P}}(\mathcal{E})$ is the $\mathbb{P}$-twist of $\mathcal{E}$ along P is explained in [48, Remark 4.4]. □

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A Orientations

In this "Appendix", we will discuss the orientations of various moduli spaces appeared in this paper. Our goal is to prove Proposition 4.1 when $\mathrm{char}(\mathbb{K}) \neq 2$. We follow the sign convention in [2]. For basic definitions, readers are referred to [2, Section 11,12] ,which we follow largely in the expositions.

A.1 Orientation operator

A linear Lagrangian brane $\Lambda^{\#} = (\Lambda, \alpha^{\#}, P^{\#})$ consists of

- a Lagrangian subspace $\Lambda \subset \mathbb{C}^n$
- a phase $\alpha^{\#} \in \mathbb{R}$ such that $e^{2\pi\sqrt{-1}\alpha^{\#}} = Det_{\Omega}(\Lambda)$
- a Pin_n-space $P^{\#}$ together with an isomorphism $P^{\#} \times_{Pin_n} \mathbb{R}^n \cong \Lambda$.

Here, Det_{Ω} is the square of the standard complex volume form on $\mathbb{C}^n$. The k-fold shift $\Lambda^{\#}[k]$ of $\Lambda^{\#}$ is given by $(\Lambda, \alpha^{\#} - k, P^{\#} \otimes \lambda^{top}(\Lambda)^{\otimes k})$, where λ^{top} is the top exterior power. For every pair of linear Lagrangian branes $(\Lambda_0^{\#}, \Lambda_1^{\#})$, one can define the index $\iota(\Lambda_0^{\#}, \Lambda_1^{\#})$ and an orientation line (i.e. a rank one $\mathbb{R}$-vector space) $o(\Lambda_0^{\#}, \Lambda_1^{\#})$.

Now, we explain how the indices and orientation lines are related to Fredholm operators. Let $S \in \mathcal{R}^{d+1}$, and $E = S \times \mathbb{C}^n$ be regarded as a trivial symplectic vector bundle over S. Let $F \subset E$ be a Lagrangian subbundle over ∂S. For each strip-like end ϵ^i, we assume $F|_{\epsilon^i(s,j)}$ is independent of s for $j = 0, 1,$. On top of that, we pick a continuous function $\alpha^{\#} : \partial S \to \mathbb{R}$ and a Pin-structure $P^{\#}$ on F such that $e^{2\pi\sqrt{-1}\alpha^{\#}(x)} = Det_{\Omega}(F_x)$ for all $x \in \partial S$. In this case, we get a pair of linear Lagrangian branes $(\Lambda^{\#}_{\xi^i,0}, \Lambda^{\#}_{\xi^i,1})$ for each puncture ξ^i, where $\Lambda^{\#}_{\xi^i,j} = (F|_{\epsilon^i(s,j)}, \alpha^{\#}(\epsilon^i(s,j)), P^{\#}_{\epsilon^i(s,j)})$ for $j = 0, 1$. We can associate a Fredholm operator $D_{S,F}$ to these data and we have [2, Proposition 11.13]

$$\operatorname{ind}(D_{S,F}) = \iota(\Lambda^{\#}_{\xi^0,0}, \Lambda^{\#}_{\xi^0,1}) - \sum_{i=1}^{d} \iota(\Lambda^{\#}_{\xi^i,0}, \Lambda^{\#}_{\xi^i,1}) \tag{A.1}$$

$$o(\Lambda^{\#}_{\xi^0,0}, \Lambda^{\#}_{\xi^0,1}) \cong \det(D_{S,F}) \otimes o(\Lambda^{\#}_{\xi^d,0}, \Lambda^{\#}_{\xi^d,1}) \otimes \cdots \otimes o(\Lambda^{\#}_{\xi^1,0}, \Lambda^{\#}_{\xi^1,1}) \tag{A.2}$$

where $\operatorname{ind}(D_{S,F})$ and $\det(D_{S,F})$ are the index and determinant line of the operator, respectively.

In the reverse direction, given $(\Lambda^{\#}_0, \Lambda^{\#}_1)$, one can pick S to be the upper half plane H and (F, α, P) such that the pair of linear Lagrangian branes at the puncture of S is $(\Lambda^{\#}_0, \Lambda^{\#}_1)$. In this special case, the operator $D_{H,F}$ has the property that $\operatorname{ind}(D_{H,F}) = \iota(\Lambda^{\#}_0, \Lambda^{\#}_1)$ and $\det(D_{H,F}) \cong o(\Lambda^{\#}_0, \Lambda^{\#}_1)$. We call $D_{H,F}$ an *orientation operator* of $(\Lambda^{\#}_0, \Lambda^{\#}_1)$.

Let ρ be a path of Lagrangian branes from $\Lambda^{\#}_1$ to $\Lambda^{\#}_1[1]$. Let S be the closed unit disk D and $(F, \alpha^{\#}, P^{\#})$ be given by $\rho(\theta)$ at the point $e^{2\pi\sqrt{-1}\theta} \in \partial S$. We denote the corresponding operator by $D_{D,\rho}$ and call it a **shift operator**. There are gluing theorems concerning how indices and determinant lines are related before and after gluing two operators at a puncture or a boundary point [2, (11.9), (11.11)]. In particular, we can glue an orientation operator of $(\Lambda^{\#}_0, \Lambda^{\#}_1)$ with $D_{D,\rho}$ at boundary points that both fibers are $\Lambda^{\#}_1$ and obtain

$$o(\Lambda^{\#}_0, \Lambda^{\#}_1) \otimes \det(D_{D,\rho}) \cong o(\Lambda^{\#}_0, \Lambda^{\#}_1[1]) \otimes \lambda^{top}(\Lambda_1) \tag{A.3}$$

By [2, Lemma 11.17], there is a canonical isomorphism $\det(D_{D,\rho}) \cong \lambda^{top}(\Lambda_1)$ so we have a canonical isomorphism

$$\sigma : o(\Lambda^{\#}_0, \Lambda^{\#}_1) \cong o(\Lambda^{\#}_0, \Lambda^{\#}_1[1]) \tag{A.4}$$

Therefore, there is a canonical isomorphism between $o(\Lambda^{\#}_0, \Lambda^{\#}_1)$ and $o(\Lambda^{\#}_0, \Lambda^{\#}_1[k])$ for all $k \in \mathbb{Z}$.

Similarly, we can consider a path of Lagrangian branes τ from $\Lambda^{\#}_0[1]$ to $\Lambda^{\#}_0$. We can use $S = D$ and τ to define an operator $D_{D,\tau}$ which we call a **front-shift operator**. In this case, we can glue an orientation operator of $(\Lambda^{\#}_0, \Lambda^{\#}_1)$ with $D_{D,\tau}$ at boundary points that both fibers are $\Lambda^{\#}_0$ and obtain

$$o(\Lambda^{\#}_0, \Lambda^{\#}_1) \otimes \det(D_{D,\tau}) \cong o(\Lambda^{\#}_0[1], \Lambda^{\#}_1) \otimes \lambda^{top}(\Lambda_0) \tag{A.5}$$

By [2, Lemma 11.17], there is a canonical isomorphism $\det(D_{D,\tau}) \cong \lambda^{top}(\Lambda_0)$ so we have a canonical isomorphism

$$\eta : o(\Lambda^{\#}_0, \Lambda^{\#}_1) \cong o(\Lambda^{\#}_0[1], \Lambda^{\#}_1) \tag{A.6}$$

A.2 Floer differential and product

Let L_i, $i = 0, 1$, be closed Lagrangian submanifolds equipped with a grading function $\theta_{L_i} : L_i \to \mathbb{R}$ (see Sect. 3.2) and a spin structure. We assume that $L_0 \pitchfork L_1$. At each point $x \in L_i$, we have a Lagrangian brane $T_x L_i^\# = (T_x L_i, \theta_{L_i}(x), Pin_x)$ inside $T_x M$ where Pin_x is the Pin_n-space determined by the spin structure on L_i. The k-fold shift $L_i[k]$ of L_i is given by applying k-fold shift to $T_x L_i^\#$ for all $x \in L_i$. For each $x \in L_0 \cap L_1$, we have a pair of Lagrangian branes $(T_x L_0^\#, T_x L_1^\#)$ inside $T_x M$. Therefore, we have the grading $|x| := \iota(T_x L_0^\#, T_x L_1^\#)$ and the orientation line $o(x) := o(T_x L_0^\#, T_x L_1^\#)$. We define $|o(x)|_{\mathbb{K}}$ to be the one dimensional $\mathbb{K}$-vector space generated by the two orientations of $o(x)$ modulo the relation that their sum is zero. An isomorphism $c : o(x) \to o(x')$ between two orientation lines can induces an isomorphism $|c|_{\mathbb{K}} : |o(x)|_{\mathbb{K}} \to |o(x')|_{\mathbb{K}}$.

Let $x_0, x_1 \in L_0 \cap L_1$ and $u : S = \mathbb{R} \times [0,1] \to M$ be a rigid element in $\mathcal{M}(x_0; x_1)$. Using the trivialization of $\Lambda_{\mathbb{C}}^{top}(M, \omega)$ together with the grading functions and spin structures on L_i, we get a trivial bundle $E = u^*TM = S \times \mathbb{C}$ and a Lagrangian subbundle F together with $(\alpha^\#, P^\#)$ over ∂S. By (A.2), we get a canonical isomorphism

$$\det(D_u) \cong o(x_0) \otimes o(x_1)^\vee \tag{A.7}$$

On the other hand, the s-translation $\mathbb{R}$-action on u induces a short exact sequence

$$\mathbb{R} \to T_u \widetilde{\mathcal{M}}(x_0; x_1) \to T_u \mathcal{M}(x_0; x_1) \tag{A.8}$$

where $\widetilde{\mathcal{M}}(x_0; x_1)$ is the moduli space of strips before modulo the $\mathbb{R}$-action. Therefore, we have an identification of the top exterior power of $T_u \widetilde{\mathcal{M}}(x_0; x_1)$ and $T_u \mathcal{M}(x_0; x_1)$, respectively. As a result, an orientation of $\mathcal{M}(x_0; x_1)$ gives an isomorphism [see (A.2)]

$$c_u : o(x_1) \to o(x_0) \tag{A.9}$$

Therefore, we can define the Floer cochain complex by

$$CF(L_0, L_1) = \oplus_{x \in L_0 \cap L_1} |o(x)|_{\mathbb{K}} \tag{A.10}$$

and the differential ∂ on $|o(x)|_{\mathbb{K}}$ is given by summing

$$\partial^{x',x} = \sum_{u \in \mathcal{M}(x';x)} |c_u|_{\mathbb{K}} : |o(x)| \to |o(x')| \tag{A.11}$$

over all x' such that $|x'| = |x| + 1$. We have $\partial^2 = 0$ [2, Section (12f)]. Similarly, given a collection of pairwisely transversally intersecting Lagrangian branes $\{L_j\}_{j=0}^d$, $x_j \in L_{j-1} \cap L_j$, $j = 1, \dots, d$, and $x_0 \in L_0 \cap L_d$, we get an isomorphism (after an orientation of $\mathcal{R}^{d+1}$ is chosen)

$$c_u : o(x_d) \otimes \cdots \otimes o(x_1) \to o(x_0) \tag{A.12}$$

for each rigid element $u \in \mathcal{M}(x_0; x_d, \dots, x_1)$, and hence a multilinear map between the relevant Floer cochain complexes. Assuming the convention of orientations in [2].

The actual A_∞ structural map $\mu^d(x_d, \ldots, x_1)$ is given by summing over all $|c_u|_{\mathbb{K}}$ with a sign twist given by $(-1)^\dagger$ (see [2, Section (12g)]), where

$$\dagger = \sum_{k=1}^{d} k|x_k| \tag{A.13}$$

In particular, $\mu^1(x) = (-1)^{|x|}\partial(x)$.

We are interested in how Floer differentials and μ^2-products (i.e. $d = 1, 2$) behave under shifts (A.4), (A.6). Let $x \in L_0 \cap L_1$ be equipped with a pair of Lagrangian branes $(T_x L_0^\#, T_x L_1^\#)$. We use $\tilde{x}$ (resp. $\bar{x}$) to denote the same intersection x being equipped with the pair of Lagrangian branes $(T_x L_0^\#, T_x L_1^\#[1])$ [resp. $(T_x L_0^\#[1], T_x L_1^\#)$]. We denote the canonical isomorphism (A.4) [resp. (A.6)] at x by $\sigma_x : o(x) \to o(\tilde{x})$ [resp. $\eta_x : o(x) \to o(\bar{x})$]. For $x_0, x_1 \in L_0 \cap L_1$ and a rigid element $u \in \mathcal{M}(x_0; x_1)$, we denote u by $\tilde{u}$ (resp. $\bar{u}$) when we regard it as an element in $\mathcal{M}(\tilde{x}_0; \tilde{x}_1)$ (resp. $\mathcal{M}(\bar{x}_0; \bar{x}_1)$). It is explained in [2, Section $12h$] that

$$\sigma_{x_0} \circ c_u = c_{\tilde{u}} \circ \sigma_{x_1} \tag{A.14}$$

It is instructive to recall the reasoning behind (A.14). Consider orientation operators D_{H,x_i}, $D_{H,\tilde{x}_i}$, the shift operators D_{D,ρ,x_i} at x_i and the linearized operator D_u defining the Floer differential. The left hand side of (A.14) $\sigma_{x_0} \circ c_u$ is obtained by first gluing D_u with D_{H,x_1}, then $D_u \# D_{H,x_1}$ with D_{D,ρ,x_0}; the right hand side $c_{\tilde{u}} \circ \sigma_{x_1}$ is obtained from gluing D_{H,x_1} with D_{D,ρ,x_1} first, and then D_u with $D_{H,x_1} \# D_{D,\rho,x_1}$.

Since the operators $(D_u \# D_{H,x_1}) \# D_{D,\rho,x_0}$ and $D_u \# (D_{H,x_1} \# D_{D,\rho,x_1})$ are homotopic (meaning the underlying path of Lagrangian subspace on the boundary are homotopic), the associativity of determinant line under gluing implies (A.14).

Similarly, we have

$$\eta_{x_0} \circ c_u = c_{\bar{u}} \circ \eta_{x_1} \tag{A.15}$$

so (A.14) and (A.15) implies that

$$\sigma \circ \partial = \partial \circ \sigma, \qquad \eta \circ \partial = \partial \circ \eta \tag{A.16}$$

Now, we consider the Floer product. Let $u \in \mathcal{M}(x_0; x_2, x_1)$ where $x_0 \in L_0 \cap L_2$ and $x_j \in L_{j-1} \cap L_j$ for $j = 1, 2$. We use u' to denote u when we regard it as an element in $\mathcal{M}(x_0; \bar{x}_2, \tilde{x}_1)$. We continue to use $D_{H,*}$ to denote an orientation operator of a Lagrangian intersection point $*$ (equipped with pair of Lagrangian branes). The gluings of D_{D,ρ,x_1} and D_{D,τ,x_2} induce the σ-operator at x_1 and η-operator at x_2, respectively. The operator $(D_u \# D_{H,x_2}) \# D_{H,x_1}$ is homotopic to $(D_{u'} \# D_{H,\bar{x}_2}) \# D_{H,\tilde{x}_1}$, and $D_{H,\bar{x}_2} \sim D_{H,x_2} \# D_{D,\tau,x_2}$, $D_{H,\tilde{x}_1} \sim D_{H,x_1} \# D_{D,\rho,x_1}$ are homotopies of operators. It implies that there is an equality

$$c_u = (-1)^{|x_1|} c_{u'} \circ (\eta_{x_2} \otimes \sigma_{x_1}) \tag{A.17}$$

where the sign $(-1)^{|x_1|}$ comes from (A.14) when moving D_{D,τ,x_2} pass D_{H,x_1}.

We abuse the notation and denote the canonical isomorphism from $CF(L_0, L_1)$ to $CF(L_0, L_1[1])$ [resp. $CF(L_0[1], L_1)$] by σ (resp. η). Denote the operator

$$(-1)^{\deg} : a \mapsto (-1)^{|a|}(a) \tag{A.18}$$

for elements of pure degree $|a|$ (and extend linearly), then $\mu^1 = \partial \circ (-1)^{deg}$. Combining (A.13), (A.16), (A.17) we have

$$\mu^1 \circ ((-1)^{\deg} \circ \sigma) = ((-1)^{\deg} \circ \sigma) \circ \mu^1 \tag{A.19}$$

$$\mu^1 \circ \eta = -\eta \circ \mu^1 \tag{A.20}$$

$$\mu^2 = \mu^2 \circ (\eta \otimes ((-1)^{\deg} \circ \sigma)) \tag{A.21}$$

Note that (A.19) is equivalent to $\mu^1 \circ \sigma = -\sigma \circ \mu^1$ but $(-1)^{\deg} \circ \sigma$ will be used later so we prefer to write in this form.

A.3 Matching orientations

We use the notations in Sect. 4. In Sect. 4.5, we proved that there are bijective identifications between the moduli

$$\mathcal{M}^{J^\tau}(\mathbf{p}'; \mathbf{p}) \simeq \mathcal{M}^{J^\tau}(c_{\mathbf{p}',\mathbf{q}}; c_{\mathbf{p},\mathbf{q}}) \tag{A.22}$$

$$\mathcal{M}^{J^\tau}(x; \mathbf{q}^\vee, \mathbf{p}) \simeq \mathcal{M}^{J^\tau}(x; c_{\mathbf{p},\mathbf{q}}) \tag{A.23}$$

$$\mathcal{M}^{J^\tau}(\mathbf{q}'^\vee; \mathbf{q}^\vee) \simeq \mathcal{M}^{J^\tau}(c_{\mathbf{p},\mathbf{q}'}; c_{\mathbf{p},\mathbf{q}}) \tag{A.24}$$

Let $\mu^{1,1}$, $\mu^{1,2}$ and $\mu^{1,3}$ be the terms of the differential of $CF(L_0, \tau_P(L_1))$ contributed by the moduli on the right hand side of (A.22), (A.23) and (A.24), respectively. In particular, we have

$$\mu^1 = \mu^{1,1} + \mu^{1,2} + \mu^{1,3} \tag{A.25}$$

and (after modulo signs)

$$\iota \circ (id \otimes \mu^1) = \mu^{1,1} \circ \iota \tag{A.26}$$

$$\iota \circ \mu^2 = \mu^{1,2} \circ \iota \tag{A.27}$$

$$\iota \circ (\mu^1 \otimes id) = \mu^{1,3} \circ \iota \tag{A.28}$$

To finish the proof of Proposition 4.1, it suffices to find a collection of isomorphisms

$$I_{\mathbf{p},\mathbf{q}} : o(\mathbf{q}^\vee) \otimes o(\mathbf{p}) \to o(c_{\mathbf{p},\mathbf{q}}) \tag{A.29}$$

for all $\mathbf{q}^\vee \otimes \mathbf{p} \in \mathcal{X}_a(C_0)$ such that

$$|I|_{\mathbb{K}} \circ (id \otimes \mu^1) = \mu^{1,1} \circ |I|_{\mathbb{K}} \tag{A.30}$$

$$|I|_{\mathbb{K}} \circ \mu^2 = \mu^{1,2} \circ |I|_{\mathbb{K}} \tag{A.31}$$

$$|I|_{\mathbb{K}} \circ (\mu^1 \otimes (-1)^{\deg -1}) = \mu^{1,3} \circ |I|_{\mathbb{K}} \tag{A.32}$$

where $I = (\bigoplus_{\mathbf{q}^\vee \otimes \mathbf{p} \in \mathcal{X}_a(C_0)} I_{\mathbf{p},\mathbf{q}}) \oplus (\bigoplus_{x \in \mathcal{X}_b(C_0)} id_{o(x)})$, and $id_{o(x)}$ is the identity morphism from $o(x)$ to $o(\iota(x)) = o(x)$ for $x \in \mathcal{X}_b(C_0)$. Notice that, the sign in (A.32) [and the absence of signs in (A.30), (A.31)] comes from the fact that (see Sect. 2.5)

$$\mu^1(\mathbf{q}^\vee \otimes \mathbf{p}) = (-1)^{|\mathbf{p}|-1}\mu^1(\mathbf{q}^\vee) \otimes \mathbf{p} + \mathbf{q}^\vee \otimes \mu^1(\mathbf{p}) + \mu^2(\mathbf{q}^\vee, \mathbf{p}) \tag{A.33}$$

In this section, we give the definition of $I_{\mathbf{p},\mathbf{q}}$ and check that (A.30), (A.31), (A.32) hold. Since the sign computation is local in nature and it is preserved under the covering map $T^*\mathbf{U} \to T^*U$, we assume that $\mathcal{E} = \mathbf{P} = S^n$.

First, we consider the case when $|\mathbf{q}^\vee| = 1$ for any $\mathbf{q}^\vee \otimes \mathbf{p} \in CF(\mathbf{P}, L_1) \otimes CF(L_0, \mathbf{P})$. In this case, we can perform a graded Lagrangian surgery (see [29] or [21]) $\mathbf{P}\#_{\mathbf{q}} T^*_{\mathbf{q}}\mathbf{P}$, which means that $\mathbf{P}\#_{\mathbf{q}} T^*_{\mathbf{q}}\mathbf{P}$ can be equipped with a grading function so that its restriction to $\mathbf{P}\backslash\{\mathbf{q}\}$ and $T^*_{\mathbf{q}}\mathbf{P}\backslash\{\mathbf{q}\}$ are the same as the grading functions on $\mathbf{P}\backslash\{\mathbf{q}\}$ and on $T^*_{\mathbf{q}}\mathbf{P}\backslash\{\mathbf{q}\}$, respectively. Moreover, all $\mathbf{P}$, $T^*_{\mathbf{q}}\mathbf{P}$ and $\mathbf{P}\#_{\mathbf{q}} T^*_{\mathbf{q}}\mathbf{P}$ are spin and the (unique) spin structure on $\mathbf{P}\#_{\mathbf{q}} T^*_{\mathbf{q}}\mathbf{P}$ restricts to the (unique) spin structure on $\mathbf{P}\backslash\{\mathbf{q}\}$ and on $T^*_{\mathbf{q}}\mathbf{P}\backslash\{\mathbf{q}\}$, respectively.

In this case, we have a canonical identification of $o(\mathbf{p})$, viewed as a subspace of $CF(T^*_{\mathbf{p}}\mathbf{P}, \mathbf{P})$ and of $CF(T^*_{\mathbf{p}}\mathbf{P}, \mathbf{P}\#_{\mathbf{q}} T^*_{\mathbf{q}}\mathbf{P})$, respectively. Moreover, $\mathbf{P}\#_{\mathbf{q}} T^*_{\mathbf{q}}\mathbf{P}$ is Hamiltonian isotopic to $\tau_{\mathbf{P}}(T^*_{\mathbf{q}}\mathbf{P})$, which sends $\mathbf{p}$ to $c_{\mathbf{p},\mathbf{q}}$, and the Hamiltonian interwines the brane structures (i.e. grading functions and spin structures on the Lagrangians). Therefore, we have an isomorphism

$$\Phi_{Ham} : o(\mathbf{p}) \cong o(c_{\mathbf{p},\mathbf{q}}) \tag{A.34}$$

from $o(\mathbf{p}) \subset CF(T^*_{\mathbf{p}}\mathbf{P}, \mathbf{P})$ to $o(c_{\mathbf{p},\mathbf{q}}) \subset CF(T^*_{\mathbf{p}}\mathbf{P}, \tau_{\mathbf{P}}(T^*_{\mathbf{q}}\mathbf{P}))$. Any choice of an isomorphism

$$\Phi_{sur} : o(\mathbf{q}^\vee) \to \mathbb{R} \tag{A.35}$$

will give us an isomorphism

$$\Phi := \Phi_{sur} \otimes \Phi_{Ham} : o(\mathbf{q}^\vee) \otimes o(\mathbf{p}) \to \mathbb{R} \otimes o(c_{\mathbf{p},\mathbf{q}}) = o(c_{\mathbf{p},\mathbf{q}}) \tag{A.36}$$

for every $\mathbf{q}^\vee \otimes \mathbf{p}$ such that $|\mathbf{q}^\vee| = 1$. We assume that a choice of Φ_{sur} is made for the moment (the actual choice will be uniquely determined by Lemma A.1).

Now, for general $\mathbf{q}^\vee \otimes \mathbf{p}$, we consider the isomorphism (see Sect. A.2)

$$\phi := \eta \otimes ((-1)^{\deg} \circ \sigma) : CF(\mathbf{P}, L_1) \otimes CF(L_0, \mathbf{P}) \to CF(\mathbf{P}[1], L_1) \otimes CF(L_0, \mathbf{P}[1]) \tag{A.37}$$

and we define $I_{\mathbf{p},\mathbf{q}}$ by

$$I_{\mathbf{p},\mathbf{q}} := \Phi \circ \phi^{1-|\mathbf{q}^\vee|} : o(\mathbf{q}^\vee) \otimes o(\mathbf{p}) \to o(c_{\mathbf{p},\mathbf{q}}) \tag{A.38}$$

Notice that $|\sigma^{1-|\mathbf{q}^\vee|}(\mathbf{p})| = |\mathbf{p}| + |\mathbf{q}^\vee| - 1 = |c_{\mathbf{p},\mathbf{q}}|$, and one should view this isomorphism as identifying $o(\mathbf{p})$ with $o((\sigma)^{1-|\mathbf{q}^\vee|}(\mathbf{p}))$ by a **sign-twisted** shift followed by identifying $o((\sigma)^{1-|\mathbf{q}^\vee|}(\mathbf{p}))$ and $o(c_{\mathbf{p},\mathbf{q}})$ by a Hamiltonian isotopy. Readers should be convinced from (A.19) that it is sensible to use the sign-twisted shift $(-1)^{\deg} \circ \sigma$.

Lemma A.1 *There is a choice of* Φ_{sur} *such that* (A.31) *holds.*

Proof To prove (A.31), we start with the case that $|\mathbf{q}^\vee| = 1$. The bijection (A.23) is obtained by the bijection $\mathcal{M}^{J^-}(\varnothing; \mathbf{q}^\vee, \mathbf{p}, x_{\mathbf{q},\mathbf{p}}) \simeq \mathcal{M}^{J^-}(\varnothing; c_{\mathbf{p},\mathbf{q}}, x_{\mathbf{q},\mathbf{p}})$. As before, we identify $o(c_{\mathbf{p},\mathbf{q}})$ with $o(\mathbf{p})$ by the Hamiltonian isotopy defining Φ_{Ham}. In this case, the linearized operator $D_{c_{\mathbf{p},\mathbf{q}},x_{\mathbf{q},\mathbf{p}}}$ corresponding to the latter moduli is homotopic to $D_{\mathbf{q}^\vee,\mathbf{p},x_{\mathbf{q},\mathbf{p}}} \# D_{H,\mathbf{q}^\vee}$, where $D_{\mathbf{q}^\vee,\mathbf{p},x_{\mathbf{q},\mathbf{p}}}$ is the linearized operator corresponding to the former moduli and $D_{H,\mathbf{q}^\vee}$ is an orientation operator of $\mathbf{q}^\vee$. The fact that these two operators are homotopic is a reflection of the fact that we can perform a graded Lagrangian surgery $\mathbf{P}\#_{\mathbf{q}} T^*_{\mathbf{q}}\mathbf{P}$ compatible with the spin structures when $|\mathbf{q}^\vee| = 1$. As a result, there is a choice of Φ_{sur} such that

$$c_u = c_{u'} \circ (\Phi_{sur} \otimes \Phi_{Ham}) : o(\mathbf{q}^\vee) \otimes o(\mathbf{p}) \to o(x) \tag{A.39}$$

where $u \in \mathcal{M}^{J^\tau}(x; \mathbf{q}^\vee, \mathbf{p})$ and $u' \in \mathcal{M}^{J^\tau}(x; c_{\mathbf{p},\mathbf{q}})$ is the element corresponding to u under the bijection (A.23) for $\tau \gg 1$, where the bijection of moduli spaces persists. We use such a choice of Φ_{sur} from now on. In particular, it means that

$$\mu^2 = \mu^{1,2} \circ |\Phi|_{\mathbb{K}} \tag{A.40}$$

for $\mathbf{q}^\vee \otimes \mathbf{p}$ such that $|\mathbf{q}^\vee| = 1$. For general $\mathbf{q}^\vee \otimes \mathbf{p}$, we use (A.21) and (A.40) to deduce that

$$|I|_{\mathbb{K}} \circ \mu^2 = |\Phi|_{\mathbb{K}} \circ \mu^2 \circ |\phi^{1-|\mathbf{q}^\vee|}|_{\mathbb{K}} = \mu^{1,2} \circ |I|_{\mathbb{K}} \tag{A.41}$$

which is exactly the desired (A.31). □

With the choice of Φ_{sur} chosen in Lemma A.1, we can now proceed and prove (A.30), (A.32).

Lemma A.2 *The Eq.* (A.30) *holds.*

Proof To show (A.30), we again first consider $\mathbf{q}^\vee \otimes \mathbf{p}$ such that $|\mathbf{q}^\vee| = 1$. Let $\mathbf{p}' \in L_0 \cap \mathbf{P}$ such that $|\mathbf{p}'| = |\mathbf{p}| + 1$. The bijection (A.22) is obtained from the bijection $\mathcal{M}^{J^-}(\mathbf{p}'; \mathbf{p}, x_{\mathbf{p}',\mathbf{p}}) \simeq \mathcal{M}^{J^-}(c_{\mathbf{p}',\mathbf{q}}; c_{\mathbf{p},\mathbf{q}}, x_{\mathbf{p}',\mathbf{p}})$. By the Hamiltonian isotopy defining Φ_{Ham}, we see that the linearized operator corresponding to the former moduli is homotopic to the linearized operator corresponding to the latter moduli. It implies that

$$\Phi_{Ham} \circ c_u = c_{u'} \circ \Phi_{Ham} : o(\mathbf{p}) \to o(c_{\mathbf{p}',\mathbf{q}}) \tag{A.42}$$

where $u \in \mathcal{M}^{J^\tau}(\mathbf{p}'; \mathbf{p})$ and $u' \in \mathcal{M}^{J^\tau}(c_{\mathbf{p}',\mathbf{q}}; c_{\mathbf{p},\mathbf{q}})$ is the element corresponding to u under the bijection (A.22). It implies that [note that $|\mathbf{p}| = |c_{\mathbf{p},\mathbf{q}}|$ and $\mu^1(a) = (-1)^{|a|}\partial(a)$, see (A.13)]

$$|\Phi_{Ham}|_{\mathbb{K}} \circ \mu^1 = \mu^{1,1} \circ |\Phi_{Ham}|_{\mathbb{K}} \tag{A.43}$$

for $\mathbf{q}^\vee \otimes \mathbf{p}$ such that $|\mathbf{q}^\vee| = 1$. It also means that, whatever isomorphism we choose for Φ_{sur}, we have

$$|\Phi|_{\mathbb{K}} \circ (id \otimes \mu^1) = \mu^{1,1} \circ |\Phi|_{\mathbb{K}} \tag{A.44}$$

For general $\mathbf{q}^\vee \otimes \mathbf{p}$, we use (A.19) and (A.44) to deduce that

$$|I|_{\mathbb{K}} \circ (id \otimes \mu^1) = |\Phi_{sur} \circ \eta^{1-|\mathbf{q}^\vee|}|_{\mathbb{K}} \otimes |\Phi_{Ham} \circ ((-1)^{\deg} \circ \sigma)^{1-|\mathbf{q}^\vee|}|_{\mathbb{K}} \circ \mu^1 \tag{A.45}$$

$$= |\Phi_{sur} \circ \eta^{1-|\mathbf{q}^\vee|}|_{\mathbb{K}} \otimes (\mu^{1,1} \circ |\Phi_{Ham} \circ ((-1)^{\deg} \circ \sigma)^{1-|\mathbf{q}^\vee|}|_{\mathbb{K}}) \tag{A.46}$$

$$= \mu^{1,1} \otimes |I|_{\mathbb{K}} \tag{A.47}$$

which is exactly the desired (A.30). □

Lemma A.3 *The Eq.* (A.32) *holds.*

Proof To prove (A.32), we appeal to an algebraic argument instead of identifying the moduli directly. Let $V_{m,n}$ be the subspace of $CF(\mathbf{P}, L_1) \otimes CF(L_0, \mathbf{P})$ generated by $o(\mathbf{q}^\vee) \otimes o(\mathbf{p})$ such that $|\mathbf{q}^\vee| = m$ and $|\mathbf{p}| = n$. The bijection (A.24) comes from the bijection $\mathcal{M}^{J^-}(\mathbf{q}'^\vee; x_{\mathbf{q},\mathbf{q}'}, \mathbf{q}^\vee) \simeq \mathcal{M}^{J^-}(c_{\mathbf{p},\mathbf{q}'}; x_{\mathbf{q},\mathbf{q}'}, c_{\mathbf{p},\mathbf{q}})$. Therefore, for each $a \in \mathbb{Z}$, there is $f(a) \in \{0, 1\}$ such that

$$|\Phi|_{\mathbb{K}} \circ (\mu^1 \otimes id)|_{V_{1,a}} = (-1)^{f(a)} \mu^{1,3} \circ |\Phi|_{\mathbb{K}}|_{V_{1,a}} \tag{A.48}$$

We remark that the existence of f follows from the fact that the sign only depends on $|\mathbf{p}|$ and $|\mathbf{q}^\vee|$ (because once $|\mathbf{p}|$ and $|\mathbf{q}^\vee|$ are determined, the local model computing the sign is determined).

By (A.20), we have $\phi \circ (\mu^1 \otimes id) = -(\mu^1 \otimes id) \circ \phi$ so we get

$$(-1)^{1-k}|\Phi \circ \phi^{1-k}|_{\mathbb{K}} \circ (\mu^1 \otimes id)|_{V_{k,a+1-k}} = (-1)^{f(a)} \mu^{1,3} \circ |\Phi \circ \phi^{1-k}|_{\mathbb{K}}|_{V_{k,a+1-k}} \tag{A.49}$$

by precomposing (A.48) by $|\phi^{1-k}|_{\mathbb{K}}$. By relabelling the subscripts, we have

$$|I|_{\mathbb{K}} \circ (\mu^1 \otimes id)|_{V_{m,n}} = (-1)^{f(m+n-1)+1-m} \mu^{1,3} \circ |I|_{\mathbb{K}} \tag{A.50}$$

The A_∞-relations on $CF(\mathbf{P}, L_1) \otimes CF(L_0, \mathbf{P})$ give

$$\mu^1 \circ \mu^2 + \mu^2 \circ (id \otimes \mu^1) + \mu^2 \circ (\mu^1 \otimes (-1)^{\deg -1}) = 0 \tag{A.51}$$

On the other hand, $CF(L_0, \tau_P(L_1))$ is a cochain complex so by considering the square of differential with input in $\mathcal{X}_a(C_1)$ and output in $\mathcal{X}_b(C_1)$ (Sect. 4.1), we get

$$\mu^1 \circ \mu^{1,2} + \mu^{1,2} \circ \mu^{1,1} + \mu^{1,2} \circ \mu^{1,3} = 0 \tag{A.52}$$

Since we have already proved (A.30) and (A.31), when we apply $|I|_{\mathbb{K}}$ to the left of (A.51) and on the right of (A.52), we get (after cancellation)

$$\mu^{1,2} \circ |I|_{\mathbb{K}} \circ (\mu^1 \otimes (-1)^{\deg -1}) = \mu^{1,2} \circ \mu^{1,3} \circ |I|_{\mathbb{K}} \tag{A.53}$$

Applying it to $V_{m,n}$ and plugging in (A.50), we have

$$(-1)^{(f(m+n-1)+1-m)+(n-1)} \mu^{1,2} \circ \mu^{1,3} \circ |I|_{\mathbb{K}} = \mu^{1,2} \circ \mu^{1,3} \circ |I|_{\mathbb{K}} \tag{A.54}$$

When $\mu^{1,2} \circ \mu^{1,3} \circ |I|_{\mathbb{K}} \neq 0$, it is possible only when $(f(m+n-1)+1-m)+(n-1)$ is even. In particular, we have $f(a) = a - 1$ modulo 2. Put it back to (A.50), we get (A.32). □

Proof of Proposition 4.1 It follows from Lemmas A.2, A.1 and A.3. □

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